Comparing Approximations for European Option Prices with Known Dividends

J oão A maro de M atos and A na L acer da
Faculdade de Economia
Universidade Nova de Lisboa

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Abstract

In this paper we discuss the properties of two approximations used to evaluate European Option written on an stock paying a discrete, known dividend in the context of a Black-Scholes economy. Using an integral representation for the exact value for this type of instrument, we study the accuracy of the different approximations using numerical methods. Regions of parameters where one approximation is better than the other are established.
1 Introduction

Certain European options written on a dividend paying stock can be easily priced in the context of a Black-Scholes economy. With either a continuous dividend yield or a discrete dividend proportional to the stock price, the Black-Scholes model can be used with simple modifications. However, when the dividend process is discrete and constant, the simplicity of the Black-Scholes model breaks down.

By a Black-Scholes economy, we understand an economy constituted by two assets: a risk-free asset with a constant rate of return and a risky asset on which the option is written. The value of the risky asset is assumed to follow a Geometrical Brownian Motion with constant drift and volatility. For such an economy there is a closed form solution, the Black-Scholes formula, to value any European option written on the risky asset. Among other things, the Black-Scholes formula depends on the initial value and on the volatility of the risky asset.

When considering that the risky asset pays a dividend $D$ at time $t_1$; a jump of size $D$ in the value process is assumed to happen at that point in time. The standard procedure for valuing European options written on such a risky asset considers the usual Black-Scholes formula where the initial price of the underlying stock is replaced by its actual value less the present value of the dividends, i.e.,

$$S_0^R = S_0 - PV(Div). \tag{1}$$

This adjustment is made to evaluate the option at any point in time before
After the payment of dividends, there is no need for any adjustment. In other words, the input in the Black-Scholes formula is the value of the continuous process

\[ S_t\] \hspace{1cm} \text{for} \hspace{1cm} t < \tau \]

The discontinuous stock price process \( S_t \) before \( \tau \) can thus be seen as the sum of two components: \( S_t = S_t^\pi + D e^{r(\tau - t)} \). There is a riskless component \( D e^{r(\tau - t)} \), corresponding to the known dividends during the life of the option and a continuous risky component \( S_t^\pi \). The riskless component, at any given time before \( \tau \), is the present value of the dividend discounted at the present at the risk-free rate. For any time after \( \tau \) until the time the option matures, the dividend will have been paid and the riskless component will no longer exist. We thus have \( S_{\tau} = S_{\tau}^\pi \) and, as pointed out by Roll (1977), the usual Black-Scholes formula is correct to evaluate the option only if \( S_t^\pi \) follows a Geometric Brownian Motion. In that case we would use in the formula \( S_0^\pi \) for the initial value together with the volatility of the process \( S_t^\pi \) followed by the risky component of the underlying asset.

The problem is that under the assumption of a Black-Scholes economy, although the risky component \( S_t^\pi \) follows a continuous process, it does not follow a Geometric Brownian Motion for times smaller than \( \tau \). Therefore, the standard procedure described just above is not valid and must be seen just as an approximation to the true value of such calls. In fact, as noticed in the early papers about option pricing (e.g., Cox and Ross, 1976; Merton, 1976a; Merton, 1976b) the correct specification of the stochastic process followed by
the value of the underlying stock is of prime importance in option valuation.

The deficiency of this standard procedure is reported in Beneder and Vorst (2001). These authors calculate the values of call options using Monte Carlo simulations and compare them with the values obtained with the approach just described. Reported errors are up to order of 9.4%. They also find that the above procedure usually undervalues the options. For that reason, Beneder and Vorst (2001) propose a method (referred to hereafter as the BV approximation) that tries to improve the standard procedure by also adjusting the volatility of the underlying asset. Their approach consists of specifying a variance of the returns that is a weighted average of an adjusted and an unadjusted variance, where the weighting depends on the timing $\xi$ of the dividend.

The objective of this paper is to compare the accuracy of the two approximation methods described above and to characterize the circumstances under which one is better than the other.

This paper is organized as follows. In Section 2 the pricing formula for an European options written on an asset paying discrete dividends is derived. Section 3 presents a discussion of the two different approximation procedures, characterizing their differences. In section 4, we characterize sufficient conditions under which the standard procedure is better than the one proposed by Beneder and Vorst. We also show sufficient conditions for the converse to happen. Section 5 concludes.
2 Valuation of European options on a stock paying discrete dividends

The uncertainty about the time evolution of the prices is represented by a complete probability space $(\Omega; F; Q^\omega)$ with an augmented filtration $F = \bigcup_{t \in [0; \omega]} F_t^\omega$ of sub-$\mathbb{Q}$-algebras of $F$ satisfying the usual conditions. Let $W_t^\omega$ denote a standard Wiener process under $Q^\omega$:

Consider the existence of a call option with maturity $T$ and strike price $K$ in the usual setting of a Black-Scholes economy, where the value of the underlying asset, $S_t$, follows a Geometric Brownian motion between any dividend payments. In other words, between dividends, the price $Q^\omega$-dynamics of $S_t$ is given by

$$dS_t = \dot{S}_t dt + \gamma S_t dW_t^\omega;$$

$W_t^\omega$ being a standard Wiener process with respect to $Q^\omega$. Then, for $S_t = s$ the arbitrage free value of the call is given by\(^1\)

$$C(t; s) = e^{-r(T-t)} E_{t,s}^Q [\gamma(S_T)];$$

(2)

where

$$\gamma(S_T) = [S_T, K]^\dagger$$

and $Q$ is a probability measure, equivalent to $Q^\omega$; and such that the $Q$-dynamics of $S_t$ is given by

$$dS_t = rS_t dt + \gamma S_t dW_t;$$

(3)

\(^1\)See, for example, Bjork (1998).
$W_t$ being a standard Wiener process with respect to $Q$: Assume now that the underlying asset pays a dividend of size $D$ at time $\zeta$ with $\zeta < T$. In this case the $Q$-dynamics of $S_t$ must be complemented with the jump condition

$$S_\zeta = S_\zeta \mid D$$

at each dividend point $\zeta$:

Solving the stochastic differential equation (3), with the jump condition (4) at the dividend point $t = \zeta$, we obtain the following rule for the stock price process at $T$

$$S_T = S_\zeta e^{\int_\zeta^T r \, dt + \frac{\sigma^2}{2} \left( (W_T - W_\zeta) + \frac{\sigma^2}{2} (W_T - W_\zeta) \right)}$$

Thus, with

$$X = S_0 e^{\int_0^\zeta r \, dt + \frac{\sigma^2}{2} \left( (W_\zeta - W_0) + \frac{\sigma^2}{2} (W_\zeta - W_0) \right)} \mid D$$

and

$$Y = e^{\int_\zeta^T r \, dt + \frac{\sigma^2}{2} \left( (W_T - W_\zeta) + \frac{\sigma^2}{2} (W_T - W_\zeta) \right)}$$

we can write

$$S_T = (X \mid D) Y$$

where $X$ and $Y$ are independent stochastic variables with density of proba-
bility given respectively by \(^2\)

\[
f(x) = \frac{1}{\sqrt{2\pi}x} \exp \left( -\frac{\ln x + \ln S_0 i - r i \frac{T}{2} \ln x}{2\sqrt{r} S_0} \right), \quad (8)
\]

\[
g(y) = \frac{1}{\sqrt{2\pi}i\sqrt{T}} \exp \left( -\frac{\ln y + r i \frac{T}{2} \ln y - r i (T + i) \frac{T}{2}}{2(T + i)\sqrt{r} y} \right), \quad (9)
\]

According to (2) and (7) the value of the call can be written as

\[
C(0; S_0) = e^{iT} E_0^T [(X \ i \ D) Y i \ K]^+ \quad (10)
\]

\[
= e^{iT} \int_0^{Z_{+1}} dx \int_0^{Z_{+1}} dy [(X \ i \ D) Y i \ K]^+.
\]

Notice that the integrand is different from zero only if \((x \ i \ D) y > K\): Since both \(x\) and \(y\) are strictly positive, this means that the integrand is different from zero only if \(y > K = (x \ i \ D)\) and \(x > D\): The value of the call can therefore be written as

\[
C(0; S_0) = e^{iT} \int_D^{Z_{+1}} dx \int_0^{Z_{+1}} dy [(X \ i \ D) Y i \ K]^+ \quad (11)
\]

Unlike the case of conventional European options, this expression cannot be integrated to provide a solution in terms of tabulated functions, such as the cumulative Normal distribution. Therefore, solving (11) will require numerical methods. Notice also that for \(D = 0\); the above expression can be integrated leading to the famous Black-Scholes expression

\[
C(0; S_0) = S_0 N (d_1) - K e^{iT} N (d_2) \quad (12)
\]

\(^2\)See Appendix A.
where \( N(\phi) \) denotes the standard Normal cumulative distribution and with
\[
d_1 = \frac{\log(S_0-K) + \left( r + \frac{1}{2} \sigma^2 \right) T}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \frac{\sigma \sqrt{T}}{
3}
\]

3 The Approximations

3.1 The Standard Procedure

The standard procedure to value European options on an underlying asset paying a discrete, constant dividend at a known date, consists in an adjustment of the current stock price by subtracting the present value of future dividends that are paid before option maturity, as in equation (1)

\[
S_t^n = S_t \cdot PV(Div) = S_t \cdot D e^{r_t(t)}
\]

for all \( t < T \). This leads to a call price similar to the Black-Scholes, except that the initial value of the stock \( S_0 \) is replaced by \( S_0 - D e^{r_T(t)} \):

\[
C(0; S_0) = S_0 \cdot D e^{r_T(t)} N(d_1) - K e^{r_T(t)} N(d_2)
\]

with
\[
d_1 = \frac{\log[(S_0 - D e^{r_T(t)})/K] + (r + \frac{1}{2} \sigma^2) T}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \frac{\sigma \sqrt{T}}{
3}
\]

The argument in using this standard procedure is that as long as \( S_t^n \) follows a stochastic exponential equation with volatility \( \frac{1}{3} \), we can think of \( S_t^n \) as being a stochastic process with the same volatility \( \frac{1}{3} \). If \( S_t^n \) follows a stochastic exponential equation, the application of the Black-Scholes formula to derive the option value is straightforward. However, this is not the correct
procedure because if $S_t$ follows the process defined in (3) together with the jump condition (4), then the risky component $S^n_t$ follows the process

$$dS^n_t = rS^n_t dt + \sqrt{\Delta} dW_t; \quad \text{for } t < \iota$$

with initial value $S^n_0 = S_{0, \iota} D e^{r\iota}$; This is not a Geometric Brownian motion for $t < \iota$; and the Black-Scholes formula (12) is no longer valid.

The proposed approximation leading to (13) consists in replacing the process $S^n_t$ by a Geometric Brownian motion $S_t$ that follows a $Q$-dynamics given by

$$dS_t = rS_t dt + \sqrt{\gamma} dW_t; \quad \text{for } t < \iota$$

$$S_0 = S_{0, \iota} D e^{r\iota}$$

Solving the above stochastic differential equation (15) with the initial condition (16), we obtain the following rule for the approximated value of the stock price process at $T$

$$S_T = i S_{0, \iota} D e^{r\iota} e^{\frac{\Delta}{2} [T + \gamma W_T]}$$

Therefore, the value of the call is given by

$$C(0; S_0) = e^{i r T} E_0^Q h S_T i K_i^*$$

that leads to the value given in equation (13).

The expected value of the stock price at time $T$ expressed in equation (10) and in equation (18) do not coincide. This is due to the value of $S_T$.
as expressed in (5) being a sum of two Log-normally distributed random variables, and thus is clearly not a Log-normally distributed random variable, as described in equation (17). The law of the processes is different and therefore the value of the call, which is the expected value of a functional of the terminal value of the stock at time $T$, will also be different. As a result thereof the value of the call obtained with this procedure is only an approximation to the real value of the call.

3.2 The BV approximation

Beneder and Vorst (2001) propose a method that tries to correct the standard procedure by also adjusting the volatility of the underlying asset. As described above, as long as $S_t$ follows a stochastic exponential equation with volatility $\sigma$, we know that $S^n_t$ does not follow the same type of law. Beneder and Vorst suggest\footnote{This suggestion is not explicitly made in their paper.} that we think of $S^n_t$ as being approximated by a process that also follows a stochastic exponential equation, however with an adjusted volatility.

The risky component described in (14) can be rewritten as

$$dS^n_t = rS^n_t dt + \frac{1}{2}\sigma^n S^n_t dW_t$$

with

$$\frac{1}{2} < \frac{1}{2}\sigma^n; \text{ for } t < \iota$$

$$\frac{1}{2}\sigma^n = \frac{1}{4} \text{ for } t \geq \iota$$

The volatility of the risky component before the dividend payment is the volatility of the stock $\frac{1}{4}$ multiplied by $S=S^n$ and the volatility after the
dividend payment equals $\frac{3}{4}$ the volatility of the stock. The BV approach consists in replacing the process $S_t^n$ by a Geometric Brownian motion $\hat{S}_t$ that follows a $Q$-dynamics given by

$$d\hat{S}_t = r\hat{S}_t dt + \gamma \hat{S}_t dW \tag{19}$$

with $\gamma$ being a constant determined at time $t = 0$ given by

$$\gamma = \frac{(\gamma S_0 - S_0)^2 \hat{c} + \gamma^2 (T - \hat{c})}{T}. \tag{21}$$

The proposed approximation consists in assuming a variance of the returns that is a weighted average of the adjusted and of the unadjusted variance, where the weighting depends on the timing $\hat{c}$ of the dividend.

This leads to a call price similar to the Black-Scholes, except that the initial value of the stock $S_0$ is replaced by $S_0 \Delta e^{r \hat{c}}$ and $\gamma$ is given by (21):

$$C(0; S_0) = \hat{S}_0 \Delta e^{r \hat{c}} \hat{C} (d_1) - K e^{rT} \hat{N} (d_2)$$

with

$$d_1 = \frac{\log [(S_0 \Delta e^{r \hat{c}}) - K] + (r + \frac{1}{2} \gamma^2)T}{\gamma \sqrt{T}}; \quad d_2 = d_1 - \gamma \sqrt{T}.$$

However, the problem is the same as in the latter section. The process $S_t^n$ is no longer an exponential stochastic equation and the application of the Black-Scholes formula is not possible.

When solving the stochastic differential equation (19), with the initial condition (20) we obtain the following rule for the stock price process at $T$}

$$S_T = \hat{S}_0 \Delta e^{r \hat{c}} e^{\frac{r}{2} \hat{c} + W_T + \gamma W_T}.$$
4 Comparing the Approximations

In this paper the price (11) has been constructed as the correct price of a call paying a discrete dividend $D$ at time $\xi$; under the assumptions of a Black-Scholes economy. Two approximations to that price were considered. As extensively discussed, both approximations for the price are based on simplifications of the process followed by the underlying stock. In spite of the expected value of the stock price at time $t = T; S_T$; coinciding with its true expected value, under both approximations discussed, the higher moments of $S_T$ do not coincide. For instance, in the appendix it is shown that the process that has the variance closest to the variance of the true process depends highly on the value of the parameters. When we compare the value of the call generated by the two different approximations with the true value of the call the best approximation, i.e., the one that gives a value closest to the true value, will also be highly dependent on the value of the parameters.

The real value of the call, the one obtained with the standard procedure and the BV approximation can be written in the form of double integrals with the same region of integration and are, respectively, given by

\[
C(0; S_0) = e^{rT} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \left( z_i K \right) h(z; x; 0; S_0; D) dz dx
\]

\[
C^\text{std}(0; S_0) = e^{rT} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \left( z_i K \right) h(z; x; 0; S_0; D e^{-r\xi}; 0; S_0) dx dz
\]

\[
C^\text{BV}(0; S_0) = e^{rT} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \int_{\mathcal{Z}_0}^{\mathcal{Z}_1} \left( z_i K \right) h(z; x; 0; S_0; D e^{-r\xi}; 0; S_0) dx dz
\]
with
\begin{align*}
\ell(x; D; S_0; \frac{3}{2}) &= \frac{1}{2^{\frac{1}{2}} x(x + D)} \exp \left( \frac{3}{2} \ln(x + D) - \frac{3}{2} \ln S_0 + r \frac{3}{2} \frac{(T - \frac{3}{2})}{\zeta} \right) \\
\ell(z; x; \frac{3}{2}) &= \frac{1}{2^{\frac{1}{2}} (T - \frac{3}{2}) \zeta} \exp \left( \frac{3}{2} \ln z + \frac{3}{2} \ln x - \frac{3}{2} \ln S_0 + r \frac{3}{2} \frac{(T - \frac{3}{2})}{\zeta} \right)
\end{align*}

and \( \% \) is as defined in (21).

In what follows we are going to analyze the circumstances under which one approximation is preferred to the other.

First of all, we will see what happens when we work with either very small or very large values of the time when the dividend is paid.

4.1 Time of the dividend \( \zeta \)

**Theorem 1** Asymptotic behavior

When \( \zeta \to T \) the limit value of the call is given by the Black-Scholes formula with \( K \) replaced by \( K + D \) and is different from the limit value of the call using any of the procedures described. On the other hand, when \( \zeta \to 0 \); the limit value of the call is given by the Black-Scholes formula with \( S_0 \) replaced by \( S_0 - D \) and is equal to the limit value of the call using any of the procedures described.

**Proof.** Consider the random variable \( Y \) defined in (6) with density of probability given by (9). As \( \zeta \) increases towards \( T \) the density \( g(y) \) converges to a degenerate distribution with mass centered in \( 1 \) and \( Y \) becomes deterministic.
As the limiting density of probability for \( S_T = (X - D) \) \( Y \) is given by

\[
\mathbb{P} \left( \frac{1}{\sqrt{2\pi} \sigma_x} \exp \left\{ \frac{\ln x - \ln S_0}{2} \frac{r - \frac{3\sigma^2}{2}}{T} \right\} \right) \geq i \geq \frac{9}{2T} := \gamma
\]

the value of the call for the limiting random variable is actually given by the Black-Scholes expression in (12) with \( K \) replaced by \( K + D \): 

\[
C(0; S_0) = S_0 N(d_1) - (K + D) e^{-rT} N(d_2)
\]

where \( N(\phi) \) denotes the standard Normal cumulative distribution and with

\[
d_1 = \frac{\log[S_0 = (K + D)] + (r + \frac{1}{2}\sigma^2) T}{\sigma \sqrt{T}} \quad d_2 = d_1 - \frac{\sigma}{2} \frac{S_0 - S_0 e^{-rT}}{D} e^{-rT}
\]

The value of the call is different from the one obtained with the standard procedure and the BV approximation. The limit value of the call using the standard procedure is given by the Black-Scholes formula with \( S_0 \) replaced by \( S_0 - D e^{-rT} \), while the one obtained with the BV approximation is given by the Black-Scholes formula with \( S_0 \) replaced by \( S_0 - D e^{-rT} \) and \( \gamma \)

\[
\gamma = \frac{S_0 - S_0 e^{-rT}}{D} e^{-rT}
\]

When \( \gamma \) converges toward 0 the limit value of the call using any of the procedures is Black-Scholes formula with \( S_0 \) replaced by \( S_0 - D \). The error of any approximation is zero. 

For a large set of values of the parameters when \( \gamma \neq 0 \) the limit value of the call is higher than the one obtained with the standard procedure. For instance, this applies when \( K = S_0 e^{rT} i \) \( D \) or \( K > S_0 e^{rT} i \) \( D \) and \( d_2 > 0 \). Therefore, using the BV procedure we may obtain a value for the call that is closer of the real value.
4.2 Value of the dividend $D$ and interest rate $r$

In what follows we are going to analyze how, for any given $\xi$; the different approximations work. When $D$ is very small or $r$ is very high the real process for the stock price approaches a stochastic exponential equation. Thus, in order to compare the real value of the call with the one obtained with the two approximations it suffices to compare the three different stochastic exponential equations for the stock price. For $\xi > t$ the three processes for the stock price coincide. However, for $\xi < t$ this is not applicable. The process for the real stock price and for the stock prices used in the standard procedure and in the BV approximation are, respectively, given by

\[
\begin{align*}
    dS_t &= rS_t dt + \frac{3}{4} S_t \left( \frac{S_t}{S_t} \right) r(\xi - t) S_t dW_t, \quad (22) \\
    dS_t &= rS_t dt + \frac{3}{4} S_t dW_t, \quad (23) \\
    dS_t &= rS_t dt + \frac{1}{4} \frac{S_t}{S_t} \left( \frac{S_t}{D e^{r(\xi - t)}} \right) \xi + (T \xi ) S_t dW_t; \quad (24)
\end{align*}
\]

The only difference in the three processes is the volatility. The volatility of the process for the stock price in (22) is always higher than the one in (23). The volatility of the process (24) is a value that is in between the volatility of the two other processes. Therefore, for small values of $D$ or high values of $r$ the process for the price used in the BV approximation is closer to the real process than the one used in the standard procedure. As a result thereof the BV approximation has a better performance.
5 Numerical results

The real value of the call does not have a closed form solution. Therefore, numerical methods were used to evaluate the value of the call and illustrate the conclusions of the previous section.

The call used in Figure [1] has an exercise price of 125 and is written on a stock that pays a discrete dividend of 5. The stock has the initial price $S_0 = 100$ and a volatility $\sigma = 0.3$. The option matures at $T = 1$.

![Figure 1:](image)

When $\sigma$ is very small the value of the call obtained with the two approximations is close from the real value. However, when $\sigma$ increases the BV approximation becomes a much better approximation.

In what follows we are going to analyze the behavior of the implicit defined function $jC_{\text{real}} - C_{\text{stand}} = jC_{\text{real}} - C_{\text{BV}} = 0$. In figures [2], [3] and [4] some sets of parameters for which one approximation performs better than the
other are presented.

The call used in Figure [2] is written on a stock that pays a discrete dividend of 5. The stock has the initial price $S_0 = 100$ and a volatility $\sigma = 0.3$. The option matures at $T = 1$.

![Diagram](image)

Figure 2: The area below each line consists of all the values of the interest rate and time of the dividend for which the BV approximation is preferred to the standard approximation. The position of each line depends on the exercise price of the call option.

The call used in Figure [3] has an exercise price of $K = 75$. It is written on a stock that pays a discrete dividend of 5 at $t = 0.8$. The stock has the initial price $S_0 = 100$ and a volatility $\sigma = 0.2$.

The call used in Figure [4] has an exercise price of $K = 100$ and matures at $T = 1$. It is written on a stock that pays a discrete dividend at $t = 0.5$. The stock has the initial price $S_0 = 100$ and a volatility $\sigma = 0.3$.

For all the pairs of parameters in the region below any line, the value of the call using the BV methodology works better than the standard procedure, i.e., the value of the call using the standard procedure is farther from the true
Figure 3: The area below the line consists of all the values of the interest rate and time of the dividend for which the BV approximation is preferred to the standard approximation. In the area above the opposite occurs.

Figure 4: The area below the line consists of all the values of the interest rate and time of the dividend for which the BV approximation is preferred to the standard approximation. In the area above the opposite occurs.
value of the call than the value obtained with the BV approximation. On the other hand, for the pairs of parameters in the region above any line it is the standard procedure that performs better. In the second figure the values of the parameters for which the two procedures have the same performance were approximated by an exponential function.

6 Conclusions

There is empirical evidence that when a discrete dividend is paid the price of the stock undergoes a jump. Assuming that the observed price of the stock is given by a stochastic exponential equation during an option's lifetime, the dividend will introduce a discontinuity into the stock price process. The price process turns out to be a discontinuous process with a jump occurring on the dividend date. As a result the Black-Scholes formula no longer applies. Therefore, in order to apply the Black-Scholes formula it has been assumed that the process that follows a stochastic exponential equation is the current price reduced by the discounted value of all the dividends to be paid during the life of the option. However, if the adjusted price follows a stochastic exponential equation, the observed stock price between dividends no longer follows the same type of process, as it is assumed in the Black-Scholes framework. Two different processes for the adjusted price were analyzed in this article.

Under the initial assumptions of the observed stock price, a valuation formula for a call option written on a dividend paying stock is derived. This formula does not have a closed form solution given in terms of tabled
functions. Using numerical methods, we were able to find regions of the parameters where better approximations for the real value of the call were obtained using one process rather than the other.

(falar de algumas conclusões)

In this paper we conclude that all the processes used until now have their shortcomings, so that one process cannot be considered to be the best approximation.

A The Distribution of X and Y

Define $X$ as

$$X = S_0 e^{\frac{h^2}{2} \mu + \frac{3}{2} W_0}$$

and $X^0$

$$X^0 = W_i W_0 > N[0, \dot{\sigma}]$$

so that the cumulative distribution for $X$ is given by

$$\Pr[X < x] = \Pr[S_0 e^{\frac{h^2}{2} \mu + \frac{3}{2} W_0} < x]$$

$$= \Pr[4X^0 < \frac{\ln y + \ln S_0}{r_i \frac{3}{2} \dot{\sigma}}]$$

$$= \frac{1}{2\pi \dot{\sigma}} \int_{-\infty}^{\ln y + \ln S_0} \frac{x}{r_i \frac{3}{2} \dot{\sigma}} e^{-\frac{x^2}{2}} dx$$

This implies that its associated density of probability is
\[ f(x) = \frac{1}{2\pi \sigma^2 x} \exp \left( \frac{-\ln x - \ln S_0 - \frac{3q^2}{2} - i\zeta}{2\sigma^2} \right), \]

i.e.,

\[ X \sim \text{LN} \ln S_0 + r i \frac{3q^2}{2} \quad \zeta; \zeta^2 \]

Proceeding in an analogous way for \( Y \), we obtain

\[ Y \sim \text{LN} \quad r i \frac{3q^2}{2} \quad (T - \zeta); (T - \zeta)^2 \]

## B Moments of the different distributions for the stock price

Notice that, being \( W_t \) an adapted Brownian motion under \( Q \), it follows that\(^4\)

\[ E^Q \left[ e^{\frac{1}{2} \sigma^2 W_t + i \zeta} \right] = e^{rt}. \quad (25) \]

We can then write the expected value of \( S_T \) as

\[ E^Q [S_T] = S_0 e^{rT + \frac{3}{2} \sigma^2 T} \quad \zeta; \zeta^2, \quad (26) \]

and its variance as

\[ \text{Var}^Q [S_T] = S_0^2 e^{2rT} \quad e^{3\sigma^2 T} \quad \zeta; \zeta^2 + (De^{rT} + 2S_0)De^{rT} e^{2rT} e^{3\sigma^2 (T - \zeta)} \quad \zeta; \zeta^2, \quad (27) \]

However it can be veriﬁed that the approximation expressed in (17) has the correct ﬁrst moment. In fact, the following holds:

\(^4\)See, for example, Björk (1998).
Proposition 2

\[ \mathbb{E}^Q S_T = \mathbb{E}^Q [S_T] \]

Proof. Using (25) to calculate the expectation of the expression in (17) leads to the value obtained in (26).

Since expected values coincide, we turn our concern to higher order moments. Consider the variance of the terminal value \( S_T \): Noticing that in the absence of arbitrage opportunities

\[ S_0 > De^{rT}; \]  

we use this fact to show that the variance of \( S_T \); to be denoted by \( \nu \); is less than the variance of \( S_T \); denoted in (27).

Proposition 3 For \( D > 0 \); then \( \nu < \nu \):

Proof. With the use of (25), we calculate

\[ \nu = S_0^2 e^{2rT} e^{\frac{3}{2}Q_T} i \left( D e^{i r \xi i} 2S_0 D e^{i r \xi e^{2rT}} e^{\frac{3}{2}Q_T} i \right); \]

Comparing this expression with (27), we obtain

\[ \zeta , \nu \nu = i D e^{i r \xi i} 2S_0 D e^{i r \xi e^{2rT} e^{\frac{3}{2}Q_T} i} 1; \]

From (28) it follows that \( \zeta > 0 \) for strictly positive \( D \).

Notice also in the proof above that the difference between the variances of the terminal value \( S_T \) disappears if \( D = 0 \):
Let $E^Q S_T$ and $v$ denote respectively the expected value and variance of $S_T$ under the probability measure $Q$ with this new volatility $\sigma$. We then have the following result for the expected value.

**Proposition 4** $E^Q S_T = E^Q S_T = E [S_T]$.

Proof. It suffices to notice that $E^Q S_T$ does not depend on $\sigma$.

For the variance of $S_T$ under $Q$ the following holds.

**Proposition 5** For $D > 0$, then $v > \mathcal{v}$.

Proof. It suffices to notice that $v$ increases with $\sigma$. The results follows from the fact that $\frac{1}{\sigma} < \frac{\sigma}{\sigma}$.

The variance of $S_T$ can also be compared with the variance of the real process $S_T$. For simplicity, let

$$I (\frac{\sigma}{\sigma}; S_0; D; r) = e^{\frac{\sigma^2}{\sqrt{\sigma^2 + \frac{s_0}{\sigma}}}} \left( e^{\frac{\sigma^2}{\sqrt{\sigma^2 + \frac{s_0}{\sigma}}}} - 1 \right).$$

(31)

Then we have the following.

**Proposition 6** For $D > 0$, then $\mathcal{v} < v$ if and only if

$$S_0^2 I (\frac{\sigma}{\sigma}; S_0; D; r) < (2S_0 i D e^{ir \sigma} + 2S_0 i D e^{ir \sigma} e^{\frac{\sigma^2}{\sqrt{\sigma^2 + \frac{s_0}{\sigma}}}} i 1).$$

Proof. With the use of (25), we calculate

$$\mathcal{v} = S_0^2 e^{2r^2} e^{(\sqrt{\sigma^2 + \frac{s_0}{\sigma}})^2} \left( e^{\frac{\sigma^2}{\sqrt{\sigma^2 + \frac{s_0}{\sigma}}}} - 1 \right) + (D e^{ir \sigma} i 2S_0) D e^{ir \sigma} e^{2r^2} e^{\sqrt{\sigma^2 + \frac{s_0}{\sigma}}}. $$
Substituting for $\frac{\mu}{\sigma}$ we obtain
\[
\varphi = S_0 e^{rT} \mu \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} (T - \tau) + \frac{\delta^2}{2} (T - \tau)^2} \varphi
\]
\[+ (D e^{rT} S_0) D e^{rT} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} (T - \tau)^2}} \varphi \]

and comparing $\varphi$ with (27), we obtain
\[
\xi, \varphi \implies \varphi = S_0 e^{rT} \frac{\mu}{\sigma} \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} (T - \tau) + \frac{\delta^2}{2} (T - \tau)^2} \varphi + \]
\[+ (D e^{rT} S_0) D e^{rT} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} (T - \tau)^2}} \varphi \]
\[= S_0 e^{rT} \frac{\mu}{\sigma} \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} (T - \tau) + \frac{\delta^2}{2} (T - \tau)^2} \varphi \]
\[+ (D e^{rT} S_0) D e^{rT} e^{\frac{\delta^2}{2} \frac{S_0}{S_0 - D e^{rT}} e^{\frac{\delta^2}{2} (T - \tau)^2}} \varphi \]

Notice that a sufficient condition to obtain $\varphi < \varphi$ is that
\[S_0 < (2S_0 \mu D e^{rT}) e^{\frac{\delta^2}{2} (T - \tau)^2} \]

The following result establishes necessary and sufficient conditions for the BV approximation to be preferred under this criterion to the standard procedure.

**Proposition 7** For $D > 0$,

\[j\varphi < j\varphi < j\varphi\]
if and only if
\[
S_0^2 \left[ I \left( \frac{3}{4} \lambda ; S_0; D; r \right) \right] < i D e^{r_\lambda} \left[ i 2S_0 e^{r_\lambda} 2e^{\frac{3}{2} \lambda} \right] \left[ i \left( \frac{3}{4} \lambda ; S_0; D; r \right) \right] i 2
\]

where 
\[
I \left( \frac{3}{4} \lambda ; S_0; D; r \right) = e^{\frac{3}{2} \lambda} \frac{S_0}{D e^{r_\lambda}} \left[ \frac{1}{2} \right] i 1.
\]

**Proof.** Since for \( D > 0; \psi > \psi \) and \( \psi < \psi \), \( j \psi \) \( \psi \) \( j \psi \) \( j \psi \) \( j \psi \) is equivalent to \( \psi \) \( \psi \) \( \psi \) \( \psi \) \( \psi \). Using the expressions obtained in the proofs of propositions 2 and 5 the conclusion is straightforward. ■

There are values of the parameters for which the variance of any of the processes used in the two approximations is at the same distance of the real variance.

Consider the values of the interest rate and the dividend \( (r_0; D_0) \) such that \( j \psi \) \( j \psi \) \( j \psi \) \( j \psi \) \( j \psi \). In this case whether we use any of the approximations it is indeterminate. The same applies for all values of the interest rate, \( r \), and the dividend, \( D \), that result in \( D e^{r_\lambda} = D_0 e^{r_\lambda} \). Graphically, the line of indetermination is given by

Let \( i \left( \frac{3}{4} \lambda ; S_0; D; r \right) \) be defined as
\[
i \left( \frac{3}{4} \lambda ; S_0; D; r \right) = S_0^2 \left[ I \left( \frac{3}{4} \lambda ; S_0; D; r \right) \right] i 1 D e^{r_\lambda} \left[ i 2S_0 e^{r_\lambda} 2e^{\frac{3}{2} \lambda} \right] \left[ i \left( \frac{3}{4} \lambda ; S_0; D; r \right) \right] i 2
\]

When \( D \) tends to zero \( \frac{\partial \left( \frac{3}{4} \lambda ; S_0; D; r \right)}{\partial D} \) approaches
\[
2S_0 e^{r_\lambda} i \frac{3}{4} \lambda + 2S_0 e^{r_\lambda} 2e^{\frac{3}{2} \lambda} i 2 \]
For all the values of parameters for which \( \frac{3\hat{i}}{2} + 2e^{\frac{3\hat{i}}{2}} \leq 0 \) the above derivative is smaller the 0 and the Vorst approximation is better than the standard procedure. Therefore, when that condition holds, for all values of the parameters in the region below the line the BV approximation works better than the standard procedure and vice-versa in the region above the line.

Just as when considering the values of the interest rate, \( r \); and the time of the dividend, \( \hat{i} \); we can find regions for the parameters for which one approximation works better than the other one. When \( r \) is very high the BV approximation works better than the standard procedure if \( \frac{3\hat{i}}{2} + 2e^{\frac{3\hat{i}}{2}} \leq 0 \). See appendix for details. When \( \hat{i} \) is small the last condition is verified for any value of \( \frac{3\hat{i}}{2} \), and a result thereof the BV approximation works better than the standard procedure.

On the other hand when \( r \) is very small (dá completamente ao contrário do que seria desejável...).
References


