A Theory of the Decision of Reverting an LBO

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1 Introduction

In this paper...

One important difference between a firm that undergoes an IPO and one that undergoes a Reverse LBO, is the fact that the latter has already been a public company, that became private when the LBO operation was performed. This means that there are two types of information about the firm that reverses the LBO: the information about the public period and the information about the private period.

In fact, in a Reverse LBO firm investors have access to information about the period where the firm was publicly traded. Examples of these additional sources of information are stock-price history, disclosures from the capital market regulator, annual reports, analysis of manager quality from the press, and opinions of market analysts. The existence of additional information implies that the uncertainty about the firm value is less in a company that makes a Reverse LBO than in a company that makes an IPO as discussed in Muscarella and Vetsuypens (1989) and in Hogan, Olson and Kish (2001).

After a LBO a firm becomes private, and the flow of information to the market will decrease drastically thereafter. However, as noted by Hogan, Olson and Kish (2001), depending on the type of debt used to finance the LBO, the market may still keep receiving some relevant information about
the firm.

During the private period after the LBO, the firm is likely to suffer modifications. In fact, the reason assumed for the existence of LBOs is that they facilitate internal restructurings and strategic modifications. If the LBO is reverted later, the firm that reappears to the market may look quite different from the one that went private earlier.

However, there is little public information about the internal restructur- ing process after the LBO, specially in comparison to the information that a public firm would have provided. This happens since in the case of a private firm, there is a much smaller number of studies and analysis than for public firms and there is no share price to aggregate and transmit information. Of course there are some sources of information as, for example, annual reports prepared by the firm. The fact that this information is prepared by the firm implies a higher risk of hidden information. Another source of information may be the perspective of going public, forcing the firm to disclose (at least partially) what happened during the private period.

In the case of a reversed LBO, outsiders have to incur in higher costs to obtain information about the private and public periods than to obtain information about the public period only. This occurs because both periods include more information than just the public period. Another relevant point is that there is less information in the public domain about the private period
than about the public period. Therefore outsiders use more resources per unit of information about the private period than per unit of information about the public period. A fundamental factor driving this result is that in a private firm there is no share price. The existence of share prices would enable many unsophisticated investors to free ride on the information transmitted by the publicly observable share price, as noted in Chemmanur and Fulghieri (1999). Hence, the evaluation costs of a firm are higher in the absence of observed share prices.

Firms reversing LBOs are evaluated by outsiders as potential investments. The precision of the evaluation technology is higher when it combines the information about both the public and private periods than when only the public period information is used.

Therefore the outsiders face a trade-off when deciding what information they should take into account when evaluating a firm: if only public period information is used, the evaluation cost is lower but the precision of the information is not as good.

2 The Model for a Reverse LBO

Consider a firm under a LBO that needs to be financed in order to undergo a new project. The firm is owned by a risk-neutral entrepreneur who holds the firm’s equity, assumed to be divided into \( m \) shares. There are two al-
ternatives presented to this firm: the project may be financed either by a Venture Capitalist (VC), in which case it would keep its private nature, or by investors, in which case it would go public and there would be a reversion of the LBO. The model to be considered has two periods. At time $t = 0$, the firm is financed by one of the two considered alternatives, and at time $t = 1$, the investment pays back.

At time $t = 0$ the firm has already finished the internal reorganization process, and has only one additional project that needs financing. If the project is not carried out, it is lost forever. This means that there is no possibility of delaying the project. At this point there is an identity between the firm and the entrepreneur. In order to finance the project, the entrepreneur must sell a certain number of additional shares to outsiders (either the VC or the investors in the public market).

Notice that the firm has several ongoing projects to which the new considered project is assumed to be added. At time $t = 1$ the set of all projects has a payoff that depends on the firm’s quality, $q$. For simplicity, we assume that the risk-free rate of return is zero.

2.1 Private Information and Investment Technology

The quality of a firm is one of two possible types: either good ($q = G$) or bad ($q = B$). The firm’s type is know by the entrepreneur but not by the out-
siders. Hence, there is asymmetry of information between the entrepreneur and the agents that are going to finance the project.

The old projects continue to generate cash flow in time $t = 1$, but they do not need new investment in $t = 0$. The cash flow generated by any project in the firm depends on several factors: the amount invested, the quality of the firm and an uncertainty component. In general, for each project, the cash flow is proportional to the investment, plus a random variable with zero mean. We take $k_q$ as the proportionality constant, reflecting the firm’s type $q$. Type good firms have larger expected cash flow than type bad firms ($k_G > k_B$).

Let $D$ denote the investment done before $t = 0$ on ongoing projects and let $\iota$ denote the amount invested in the new project at $t = 0$. We assume that there is a critical value $I$ for the amount invested above which no additional value is created. Thus, no entrepreneur will invest an amount larger than $I$. Furthermore, let $\tilde{e}_D$ and $\tilde{e}_I$ denote the mean-zero random variables modeling the uncertainty in the old and new investments respectively.

The time $t = 1$ cash flow from the firm is thus given by the following investment technology, that is known by the outsiders:

$$v_q(\iota) = k_q \left[ D + \iota \right] + \tilde{e}_D + \tilde{e}_I, \text{ for } \iota < I; \quad v_q(\iota) = k_q \left[ D + I \right] + \tilde{e}_D + \tilde{e}_I, \text{ for } \iota \geq I$$

where $\tilde{e}_M \sim (0, \sigma^2_{\tilde{e}_M}), \tilde{e}_I \sim (0, \sigma^2_{\tilde{e}_I}), q \in \{G, B\}$, and

$$k_G > k_B > 1.$$
The variance of the firm’s cash flow depend on the variances of the cash flows of the old and new projects ($\sigma^2_D$ and $\sigma^2_I$ respectively) and on the covariance between them. Let $\text{cov}(\tilde{e}_D, \tilde{e}_I) \equiv \sigma_{DI}$ denote the covariance between the cash flows of the old and new projects. Neither the entrepreneur nor the outsiders know the impact of the uncertainty on the project cash flow. They only know the uncertainty characteristics. This means that, even the entrepreneur does not know the exact cash flow at time $t = 1$.

We assume that the new firm project has sufficiently large net present value (i.e., $k_G$ and $k_B$ are large enough), in such a way that the entrepreneur has always profit in investing $I$. At the full investment level, the entrepreneur’s expectation at time $t = 0$ about the future cash flow at time $t = 1$ is

$$V_G = k_G(D + I) \quad (1)$$

for the good type and

$$V_B = k_B(D + I) \quad (2)$$

for the bad type.

### 2.2 The Outsiders’ Evaluation Technology

Following Chemmanur and Fulghieri (1999), an essential assumption is that the outsiders do not know the type of the firm that asks for financing. However, they can make an evaluation of the firm’s quality at some cost. There
are two types of information available about the firm: information about the public period of the firm and information about the private period of the firm. The outsider has a cost $c_p$ if the quality of the firm is evaluated based only on information about the public period. If the outsider uses the full information (public plus private period), must therefore pay a higher cost $c_f > c_p$.  

$$c_f > c_p.$$  

The result of the evaluation process is denoted by $e$, and may generate two results: good ($e = g$) or bad ($e = b$). The evaluation is denoted by $e_f$ or $e_p$ depending on whether it is obtained with full or simply with public information respectively. The accuracy of an evaluation is not necessarily complete. One may evaluate a firm, and still not be sure that the evaluation is correct. For a given information set, one is more or less sure of the evaluation accuracy depending on the evaluation technology used. The evaluation technology is assumed to be the same for all outsiders. We model the precision of such technology as depending on the type of information used:

$$\Pr (e_p = g_p \mid q = G) = 1. \Pr (e_p = g_p \mid q = B) = y. 0 < y < 1.$$  

$$\Pr (e_f = g_f \mid q = G) = 1. \Pr (e_f = g_f \mid q = B) = x. 0 < x < 1.$$  

In particular, we assume that a good firm gets always a good evaluation. A bad firm, however, may get either a bad or a good evaluation. Naturally we assume that the probability that the evaluation technology makes an
error is higher if only public information is used \((x < y)\). The precision of the evaluation technology increases as \(y\) and/or \(x\) decrease. Moreover, we assume that when the investors produce information about a type bad firm, a fraction \(y\) of these investors obtain good evaluations, and a fraction \((1 - y)\) obtain bad evaluations.

The outsiders’ evaluation costs, \(c_p\) and \(c_f\), depend on two important factors. First, these costs depend on the amount of information available in the market place about the firm and its management. Hence, the evaluation costs decrease as the firm becomes older and more information about the firm is accessible to everybody. Second, firms belonging to industries intrinsically more difficult to evaluate, have larger evaluation costs. The first factor is related to information availability, whereas the second factor is related to the difficulty in processing the information.

2.3 The Venture Capitalist

We assume that there exist one risk-averse VC that can finance in full the project\(^1\). The VC is assumed to invest a large portion of his/her wealth in the firm, meaning that the VC is not sufficiently diversified. Hence, the VC will demand a nondiversification premium from the investment in the firm.\(^2\)

\(^1\)For simplicity, we assume that only one VC finances the firm but it is possible that many VC finance, simultaneously, the firm without affecting our results.

\(^2\)As noted by Chemmanur and Fulghieri (1999), an alternative modelization, that produces fundamentally the same results, is to assume that the VC has bargaining power relative to the entrepreneur, which enables him to extract a fraction of the net present value of the firm’s project. The numerous investors in the public market have less bar-
We assume that the capitalist invests the wealth only in the project and in the risk-free asset. Because the VC is assumed to be risk-averse, his/her objective is to maximize the utility resulting from the cash flow received at time $t = 1$. The utility function is assumed to be of the form:

$$U(W) = \mu_w - \rho \sigma_w^2,$$  \hspace{1cm} (4)

where $\mu_w$ is the mean and $\sigma_w^2$ is the variance of the wealth at time $t = 1$, and $\rho$ the coefficient of risk-aversion.

The entrepreneur offers the VC one of three possible contracts. The first type of contract is an unconditional price contract, where the VC does not conduct an evaluation of the firm. The second type is a contract where the VC makes the evaluation only with information about the public period of the firm. Finally, the third type is a contract where the VC makes the firm evaluation with full information. In the second and third contracts the price paid by each share, and fixed by the entrepreneur in the contract, depends on the outcome of the evaluation.

In response to the entrepreneur’s offer, the VC may reject or accept the contract. The VC will accept the contract if and only if the entrepreneur offer leaves him at least as well off, in terms of utility, as investing in the risk-free asset.
2.4 Investor Strategies in Public Offers

Let us now consider the case where the firm goes public, reverting the LBO, in order to finance the new project. It is assumed that the potential investors who are financing the new project buy zero or one share of the new firm in the public offer market\(^3\). This means that each investor holds only a small fraction of the firm equity. Thus, there is no loss of generality in assuming that the investors are risk-neutral\(^4\). Since the investors are assumed to be risk neutral, their objective is to maximize the expected value of the cash flow that goes to them at time \(t = 1\). As in the case of the VC, it is assumed that the investors’ wealth not invested in the firm is invested in the risk-free asset.

The investors have four alternatives: (1) invest all wealth in the risk free asset; (2) bid for shares in the public offer with no information\(^5\); (3) evaluate the firm with information about the public period, investing in the offer if the evaluation is good or investing in the risk-free asset if the evaluation is bad\(^6\); (4) evaluate the firm with full information, investing in the offer if the

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\(^3\)We assume that all the investors have enough wealth to invest in the firm.

\(^4\)As Chemmanur and Fulghieri (1999) explain, the results will be qualitatively unchanged if the investor was also risk-averse. The crucial difference between the VC and the investor is that the latter has a greater share of his wealth tied to the firm, so either he demands a higher nondiversification premium or has more bargaining power.

\(^5\)As argued by Chemmanur and Fulghieri (1999) if there is an investment bank that gives information to the investors about the firm, this information will be used by all investors (even those that are uninformed). The existence of an investment bank can reduce but not eliminate the information asymmetry between insiders and outsiders.

\(^6\)In equilibrium there is a waste when investing in a type bad firm. This implies that for an investor it is never optimal to invest in a firm when the evaluation is bad, because
evaluation is good or investing in the risk-free asset if the evaluation is bad.

Given a public offer, after the definition of the share price and the number of shares to be sold, the investor chooses among the four alternatives described above. In equilibrium, the investor is indifferent between producing full information and producing information about the public period. We assume that a fraction $\alpha$ of the participants in the public offer produces full information, and a fraction $1 - \alpha$ produces information about the public period.\footnote{This result from the fact that, each investor follow a randomized strategy (and not pure strategies) that consiste in producing full information or public information with some probability.}

3 Equilibrium

The decision about how to finance the new project may be seen as a game where the players are the entrepreneur and the outsiders, namely the VC and the investors. In this section we are going to study the Perfect Bayesian Equilibrium of this game. The equilibrium consists of (i) a choice of financing method by the entrepreneur at time $t = 0$, and a joint about the prices and the number of shares to be offered to outsiders in each case; (ii) the decision of the VC about whether or not to invest in the firm, if the entrepreneur chooses to use the private equity; (iii) and, if the firm goes public, the decisions of each investor about: (a) whether or not to participate in the public offer; (b)
the probability of producing full information and the probability of producing information only about the public period.

In order to characterize a Perfect Bayesian Equilibrium, each of the above choices must fulfil the following conditions: (i) the choices of each agent maximize the agent’s objective, given the equilibrium beliefs and choices of others (sequential rationality); (ii) the beliefs of all players are consistent with the equilibrium choices of others and, along the equilibrium path, these beliefs are formed using Bayes’ rule (beliefs consistency); (iii) deviations from equilibrium strategies satisfying the former conditions by any agent is suboptimal.

We are going to study the pooling equilibrium with information production by the VC and the investors, because we are interested in the impact of information production in the Reverse LBO decision. However, there may exist other equilibria, depending on the parameters’ values. First, we can have a separating equilibrium, where different types of firms have different behaviors in equilibrium, therefore revealing their type, implying that there is no information production in equilibrium. Second, we can have a pooling equilibrium without information production, where either the VC, or investors, or both, invest in the firm without conducting the firm evaluation.

A separating equilibrium can exist, for example, as a limiting case of the pooling equilibrium with information production. If the pooling cost to the type bad firm is very high, all the type bad firms choose the low price and all
good firms choose the high price. Because the firms reveal their type through their actions, there is no information production by outsiders. However, if the investors do not produce information, the type bad firm is free to pool with the type good firms. Hence, such situation does not constitute an equilibrium.

A pooling equilibrium without information production occurs, for example, if the evaluation costs, \( c_p \) and \( c_f \), are very large (or if the precision of the evaluation technology is very low). In this situation, it does not compensate to the VC or to the investors to produce information. This implies that a type bad firm does not have costs in pooling with the type good firm, and so there exists a pooling equilibrium.

Given the out-of-equilibrium beliefs, it follows that with information production there is no separating equilibrium in the financing alternatives (see proof of proposition 5). In other words, the two types of entrepreneurs choose always the same financing source.

### 3.1 Private Equity Financing

If the entrepreneur decides to use private equity financing, then the entrepreneur chooses the kind of financing contract to offer the VC (contract with no information, contract with information about the public period or contract with full information), as well as the price of the equity and the
number of shares to be offered in each contract.

If the entrepreneur decides to offer the VC a contract with information production (with full information or with information about the public period), the VC makes the evaluation with the contracted information. The entrepreneur commits to compensate the VC for the cost incurred in the evaluation, independently of the implementation of the financing contract. The entrepreneur then observes the outcome of the evaluation, and decides on the fraction of equity to offer to the VC. The VC decides then whether to accept or reject the contract. In what follows we model this procedure.

3.1.1 A contract with public period information

Let $\phi$ denote the proportion of good firms in the market. Therefore, $\phi$ reflects the initial belief about the probability that a firm’s type is good under no additional information. If the VC conducts a costly evaluation of the firm, taking into account only the information about the public period, his/her initial belief will be updated using the Bayes’ rule:

$$\Pr(q = G | e_p = g_p) = \frac{\Pr(q = G \cap e_p = g_p)}{\Pr(e_p = g_p)} = \frac{\phi}{\phi + y(1 - \phi)} > \phi \quad (5)$$

$$\Pr(q = G | e_p = b_p) = 0$$

Notice that the result of the evaluation is analyzed by the VC taking into account that the evaluation was done only with public information (is for that we have the index $p$). This occurs because the VC know that the
precision of the evaluation technology depends on the information used.

The probability that a firm is of good type and, simultaneously, is evaluated as good, \( \Pr(q = G \cap e_p = g_p) = \phi \), is equal to the prior probability assessment, \( \phi \), of the firm being of type good. The probability that the evaluation is good, \( \Pr(e_p = g_p) \), is equal to the probability of a right evaluation, \( \phi \) - that occurs when the firm is good - plus the probability of a wrong evaluation, \( y(1 - \phi) \) - when the firm is bad and there is a wrong evaluation.

If the evaluation is bad, there is no possibility of the firm to be good, because we assumed that \( \Pr(q = G \mid e_p = b_p) = 0 \), meaning that all good firms get good evaluations.

We denote the VC’s expectation of firm’s time 1 cash flow, at full investment level \( I \), conditional on the outcome of this evaluation by

\[
V_{e_p} = E[k_q(D + I) \mid e_p], \text{ for } e_p \in (g_p, b_p)
\]

Notice that this expectation depends on the information used in the evaluation of the firm. We have

\[
Var[k_q(D + I) \mid e_p] = \sigma_{e_p}^2 \text{ for } e_p \in (g_p, b_p).
\]

This variance reflects an uncertainty that is information-based, since it arises from de-facto that the VC does not know the firm’s type. Given that all bad firms get bad evaluations with probability 1, we also conclude that

\[
V_{b_p} = V_B = k_B(D + I) \text{ and } \sigma_{b_p}^2 = 0.
\]
In this section we study the case where the evaluation is based on information about the public period of the firm. If the evaluation is good, let $s_{gp}^{*}$ denote the lowest share of the firm’s equity that the VC accepts in compensation for contributing to the financing of the firm with $I$. Let $s_{bp}^{*}$ denote a similar number for the case where the evaluation is bad. These fractions of equity must leave the VC indifferent between investing in the firm and investing in the risk-free asset.

Consider that the firm is evaluated as good. In that case, if the VC has invested in the firm, he/she will hold a fraction $s_{gp}$ of the firm. Hence, his/her wealth at $t = 1$ is a random variable with mean $\mu_w = s_{gp} V_{gp}$ and variance $\sigma_w^2 = s_{gp}^2 (\sigma_{gp}^2 + \sigma_{\xi D}^2 + \sigma_{\xi I}^2 + 2 \sigma_{DI})$. Thus, from expression (4) the VC utility at $t = 1$ is:

$$s_{gp} V_{gp} - \rho \sigma_{gp}^2 (\sigma_{gp}^2 + \sigma_{\xi D}^2 + \sigma_{\xi I}^2 + 2 \sigma_{DI}).$$

If the VC had alternatively invested in the risk-free asset, his/her utility at $t = 1$ would be $I + c_p$, since the rate of return of the risk-free asset is assumed to be 0. Notice that the total investment made by the VC in this case is the sum of the amount $I$ invested directly in the project, plus the evaluation costs $c_p$. Hence, $s_{gp}^{*}$ must satisfy

$$s_{gp} V_{gp} - \rho \sigma_{gp}^2 (\sigma_{gp}^2 + \sigma_{\xi D}^2 + \sigma_{\xi I}^2 + 2 \sigma_{DI}) = I + c_p. \quad (6)$$
Similarly, if the evaluation is bad, $s_{b_p}^*$ must satisfy

$$s_{b_p} V_{b_p} - \rho s_{b_p}^2 \left( \sigma_{\varepsilon D}^2 + \sigma_{\varepsilon I}^2 + 2\sigma_{DI} \right) = I + c_p$$

(7)

Furthermore, notice that, since all firms that get a bad evaluation are bad firms with probability 1, there is no information-based uncertainty if the evaluation is bad.

### 3.1.2 A contract with full information

If the evaluation is made with full information, the VC will update the probability that the firm is good, using Bayes’ rule:

$$\Pr(q = G \mid e_f = g_f) = \frac{\phi}{\phi + x(1 - \phi)} > \phi, \quad (8)$$

$$\Pr(q = G \mid e_f = b_f) = 0.$$  

Notice that, since $x < y$, it follows from equations (5) and (8) that

$$\Pr(q = G \mid e_f = g_f) > \Pr(q = G \mid e_p = g_p) > \phi.$$  

(9)

The VC expectation of the firm’s cash flow at $t = 1$, at full investment level $I$, conditional on the outcome of this evaluation is

$$V_{e_f} = E[k_q(D + I) \mid e_f], \text{ for } e_f \in (g_f, b_f).$$

The information-based uncertainty is characterized by the variance

$$Var[k_q(D + I) \mid e_f] = \sigma_{e_f}^2 \text{ for } e_f \in (g_f, b_f).$$
Since all firms that get bad evaluation are bad firms with probability 1, we know that $V_{bf} = V_B$ and $\sigma_{bf}^2 = 0$.

Now we are considering the case where the evaluation is based on full information. If the evaluation is good, let $s^*_g$ denote the lowest share of the firm’s equity that the VC accepts in compensation for contributing to the financing of the firm with $I$. Let $s^*_b$ denote a similar number for the case where the evaluation is bad. Proceeding as in the former section, these fractions must satisfy the following conditions:

$$s_g V_{gf} - \rho s_g^2 (\sigma_{gf}^2 + \sigma_{\varepsilon D}^2 + \sigma_{\varepsilon I}^2 + 2\sigma_{DI}) = I + c_f$$  \hspace{1cm} (10)$$

$$s_b V_{bf} - \rho s_b^2 (\sigma_{\varepsilon D}^2 + \sigma_{\varepsilon I}^2 + 2\sigma_{DI}) = I + c_f.$$  \hspace{1cm} (11)

### 3.1.3 An unconditional price contract

If the VC does not conduct a firm evaluation, the firm’s equity will be priced using only the prior probability $\phi$. The VC’s expectation about the firm’s cash flow at $t = 1$, at full investment level, under a unconditional price contract is

$$V_u = E[k_q(D + I) \mid \phi],$$  \hspace{1cm} (12)$$

and the variance of this cash flow is

$$Var [k_q(D + I) \mid \phi] = \sigma_u^2.$$  

Let $s^*_u$ be the minimum fraction of the firm’s equity that the VC will
accept in return for the investment $I$, when he does not conduct a firm evaluation. This fraction must satisfy the following condition:

$$s_u V_u - \rho s_u^2 (\sigma_u^2 + \sigma_{\epsilon D}^2 + \sigma_{\epsilon I}^2 + 2\sigma_{D I}) = I.$$  \hspace{1cm} (13)

Notice that, as opposed to the former cases, the evaluation costs do not appear in the right hand side of the above equation, since in this case there is no evaluation of the firm.

### 3.1.4 Contract’s choice and type of entrepreneur.

Given the rationality of the entrepreneurs, firms will choose to offer contracts that enable the new project to be financed, minimizing the fraction of equity given in exchange to the VC.

From this principle, it follows that in the case of good firms, a contract with full information is offered if and only if $s_{gf}^* < s_u^*$ and $s_{gf}^* < s_{gp}^*$. Similarly, a good entrepreneur offers a contract with information only about the public period if and only if $s_{gp}^* < s_u^*$ and $s_{gp}^* < s_{gf}^*$. Finally, this entrepreneur chooses to offer an unconditional price contract if and only if $s_u^* < s_{gp}^*$ and $s_u^* < s_{gf}^*$. Notice that in the case of a good firm, the entrepreneur knows for sure the value $s_{gf}^*$, since the firm will be always evaluated as good. This certainty does not occur in the case of a bad firm.

It also follows for the case of firms evaluated as good that $V_{gf} > V_{gp}$ >
$V_u$. This result comes from equation (9) together with the fact that $k_G > k_B$, and the equations defining $V_{gf}, V_{gp}$ and $V_u$. Similarly we have $\sigma^2_{gf} < \sigma^2_{gp} < \sigma^2_u$. This means that the uncertainty about the firm type is lower if we obtain a good evaluation (either with full information or with public period information) than if we do not make any evaluation. This altogether means that, the disadvantage of contracts with information production is the evaluation cost that the VC has to support. Hence, good entrepreneurs choose to offer contracts with full information only if the evaluation costs are not too high. In the limiting case where $c_f = c_p = 0$, we have $s^*_{gf} < s^*_u$ and $s^*_{gf} < s^*_{gp}$, as follows from equations (6),(10) and (13).

In the case of bad firms, entrepreneurs can choose between pooling or separating from the good firms. The pooling situation enables the type bad firm to sell the equity at a high price. If a type bad firm chooses to separate, the best deviating contract is the one where the type bad firm offers an unconditional price contract in conditions where the type good firm would offer a contract with information (either with full information or with information only about the public period). Such deviating contract presents two advantages to the bad firm: first, the bad type is identified as bad even without evaluation$^8$; second, there is no information-based uncertainty.

The offer of an unconditional price contract in this situation dominates

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$^8$ This happens given the out-of-equilibrium beliefs, to be described below.
the contract with no information where the investors cannot distinguish, without evaluation, between the two types of entrepreneurs. In the latter case the investors will have information-based uncertainty, and will therefore demand a higher equity fraction. With the former dominating contract (the best deviating contract) the minimum fraction to be offered to the VC is $s_B^*$, satisfying

$$ s_B V_B - \rho s_B^2 (\sigma_{\epsilon D}^2 + \sigma_{\epsilon I}^2 + 2\sigma_{DI}) = I \quad (14) $$

Notice that the VC knows with certainty that the firm’s type is bad. This implies that there is no information-based uncertainty, and therefore $V_B$ is used.

Suppose now that a bad entrepreneur chooses to pool with good firms.

Consider first the case where good firms offer a contract with information about the public period. Then, the bad firm will have a good evaluation with probability $y$, and a bad evaluation with probability $(1-y)$. Thus, the expected fraction that the firm has to give up to the VC is $ys_{gp}^* + (1-y)s_{bp}^*$, where $s_{gp}^*$ satisfies equation (6), and $s_{bp}^*$ satisfies equation (7).

Consider now the situation where good firms offer a contract with full information. Then, the bad firm will have a good evaluation with probability $x$, and a bad evaluation with probability $(1-x)$. Thus, the expected fraction that the bad firm has to give up to the VC is $xs_{gf}^* + (1-x)s_{bf}^*$, where $s_{gf}^*$ satisfies equation (10), and $s_{bf}^*$ satisfies equation (11).
A bad entrepreneur will pool with good firms only if the expected fraction that the bad firm has to give up to the VC is less than $s^*_B$. If good firms offer contracts only with public period information, bad entrepreneur will choose to pool only if $ys^*_g + (1 - y)s^*_b < s^*_B$. On the other hand, if good firms offer contracts with full information, bad entrepreneurs will pool only if $xs^*_g + (1 - x)s^*_b < s^*_B$.

In what follows we shall provide sufficient conditions for a pooling equilibrium. It is enough that the VC is not too risk-averse and that the evaluation costs with full information are not too high for a pooling equilibrium with full information contract to hold. The consistency of the equilibrium characterized by such contracts results from an assumption about out-of-equilibrium beliefs: If any entrepreneur offers the VC a contract with other parameters than those specified in the equilibrium contract, the VC infers with certainty that the firm’s type is bad.9

**Proposition 1** There is a critical value $\rho_V$ for the coefficient of risk aversion $\rho$ of the VC and a critical value $c_{f_V}$ of the evaluation cost with full information such that if $\rho < \rho_V$ and $c_f < c_{f_V}$, then there is a pooling equilibrium where (i) both good and bad firms offer a contract to the VC involving full information production and the equity fractions $\{s^*_g, s^*_b\}$ and (ii) the VC accepts the contract.

9See the proof of the Proposition in the Appendix.
The content of this Proposition is quite intuitive. Each entrepreneur will offer the VC the minimum equity share that will ensure the financing and both good and bad entrepreneurs will be financed in full. However, if the VC is too risk averse, he/she may demand an extreme remuneration that invalidates the financing process. Our result characterizes an upper bound on the coefficient of risk aversion that avoids such situations.

Good entrepreneurs have an advantage choosing contracts with full information because they will always be identified as good. If that is the case, bad entrepreneurs mimic good entrepreneurs offering a contract with full information. This may induce the VC in mistake, and allow the shares to be sold at a high price. Since bad entrepreneurs may be identified as such, their financing conditions are worse on average than those of good entrepreneurs. On the other hand, both good and bad entrepreneurs have an advantage offering a contract with no information only if the evaluation costs with full information are very high. The above result characterizes an upper bound on $c_f$ such that contracts with no information are ruled out.

In what follows we characterize how, in equilibrium, the fraction of equity that has to be offered to the VC in compensation for the firm financing depends on various parameters of the model.

**Proposition 2 (Comparative Statics)**  
In equilibrium, the fraction of equity that has to be offered to the VC increases with (a) an increase in the
cost of evaluating the firm, \( c_f \); (b) an increase in the VC’s coefficient of risk aversion, \( \rho \); (c) an increase in the risk of old and/or new firm’s projects, \( \sigma_{\varepsilon_D}^2 \) and \( \sigma_{\varepsilon_M}^2 \); (d) an increase in the covariance between the cash flows of the old and new projects, \( \sigma_{DI} \); (e) a decrease in the investment done by the firm in the past, \( D \), and (f) a decrease in the productivity of the firms, \( k_q, q = G, B \).

Finally, the impact of an increase of \( I \) on the VC’s fraction of equity is ambiguous.

Let us provide the intuition for such results. Regarding (a), if \( c_f \) increases, the VC will have to support larger evaluation costs, and a corresponding larger fraction of equity will be required to finance the firm.

Result (b) follows from the fact that an increase in the risk-aversion coefficient leads the VC to demand a higher compensation to support the same risk to finance the firm.

An increase in the risk of the old or the new projects would reduce the utility of the risk-averse VC. This would induce the VC to demand a higher compensation, leading to result (c).

For the same level of risk of the old and the new projects, an increase in the covariance between the cash flows of the old and new projects, would increase the global risk of the firm, causing the VC to demand a higher equity to finance the firm. This is result (d).

With respect to (e), if the investment done in the past were larger, then
the cash flow of the firm at \( t = 1 \) would increase. This would have occurred without the need of any additional investment from outsiders at \( t = 0 \), implying that the VC would demand a lower fraction of the equity, because the VC would already benefit from the higher investment done in the past.

Result (f) is explained as follows. If the firms were more productive, they would originate higher cash flows with the same investment. Then, because firms would have a higher value, the fraction demanded by the VC would be smaller.

Finally, the ambiguity about the impact of an increase in the level of investments in new projects is explained as follows. The VC demands a higher equity fraction, because the amount invested is assumed to be larger. A higher investment, however, produces a higher cash flow at time \( t = 1 \), leading the same fraction of equity to correspond now to a higher value. If the second effect dominates the first, we should have a decrease in the VC’s equity share. This happens if the average gain of the new investment is not much higher than the marginal gain of the new investment. In other words, for the second effect to dominate the first, it suffices that the investment in the new project, as a proportion of the investment done in the old projects, is high enough.
3.2 Public Financing

We are now going to study the equilibrium in the new issue market. If a firm chooses to issue new shares to finance the new project, a certain number of shares is offered at a fixed price. Once more, the firm’s objective is to raise the amount $I$, giving up the lowest possible share of equity\(^\text{10}\). After observing the firms’ actions, a large number of investors evaluate the firm and decides whether to invest or not. The next proposition characterizes the equilibrium in the new issue market.

**Proposition 3** There is an equilibrium in the new issues market characterized by the following: (a) Good firms issue and sell $n_H$ shares, each at a price $p_H$, raising a total amount $I$ for investment; (b) Bad firms pool with probability $\beta$ ($0 < \beta \leq 1$), with the good firms by offering $n_H$ shares at the price $p_H$. Only a number $zn_H$ of those shares are bought by investors in equilibrium ($0 < z < 1$), thus raising only an amount $zI$; with probability $(1 - \beta)$, bad firms separate from good firms, by offering and selling $n_L$ shares at a lower price $p_L$ ($n_L > n_H, p_H > p_L$), raising the entire amount $I$ for investment; (c) A fraction $\alpha$ of the investors in the public market produce information about the public period, bidding for a share if and only if they get a good evaluation; the remaining fraction $(1 - \alpha)$ produce full information, bidding for a

\(^{10}\)We assume that, if the firm cannot sell all the offered shares, the going public operation still goes forward.
share if and only if they get a good evaluation. A necessary condition for the equilibrium to exist is that (i) the difference between the costs of obtaining the two types of information is not to high, i.e.,

\[ c_f - c_p < (1 - \phi)(y - x) \left( \frac{1}{n_H} - \frac{(k_B - 1)}{m} \right) I = d_s; \]

(ii) the cost of producing full information is sufficiently higher than the cost of producing public period information, i.e.,

\[ \frac{c_f}{c_p} \geq \frac{(1 - x)}{(1 - y)} > 1; \]  \hspace{1cm} (15)

(iii) good firm’s productivity is sufficiently high, i.e., \( k_G > k_{\overline{G}} \). (\( n_H \) and \( k_{\overline{G}} \) are defined in the appendix)

In the above equilibrium, both types of firms can choose between selling a large number of shares (\( n_L \)) at a low price (\( p_L \)) or selling a small number of shares (\( n_H \)) at a high price (\( p_H \)). Naturally, the best solution for the firm is the last one. The above equilibrium is supported by the belief that outsiders infer that any firm setting a price other than \( p_H \) or \( p_L \), or offering a number of shares other than \( n_H \) (at price \( p_H \)) or \( n_L \) (at price \( p_L \)) is certainly a type bad firm.

In what follows we analyse the content of the equilibrium described above.

First, consider the behavior of good firms. Good firms will always choose to sell a small number of shares at a high price, in order to obtain the full
amount \( I \) to invest. This occurs because the firm knows that all investors that make his evaluation obtain a good evaluation.

Second, let us characterize the behavior of bad firms. If bad firms set also a high price, they will not be able to raise the entire amount of funds, \( I \), because some investors will get bad evaluations, and consequently will not invest in the firm. This causes a decrease in value for the firm. The proportion of the offered shares that are sold by a bad firm that chooses to pool with good firms is

\[
z_{nH} = N \left[ (1 - \alpha)y + \alpha x \right]
\]

Finally, we consider the behavior of the investors. Given the total number \( N \) of participants (potential investors), only a fraction invests (buy one share) with full information (\( \alpha \)), and the remaining fraction invests with information about the public period \( (1 - \alpha) \). A proportion \( x \) of investors with full information get good evaluations and invest in the reverse LBO. In the same way, a proportion \( y \) of investors with public period information get good evaluations, and buy firm shares. Note that in equilibrium uninformed investors do not exist.
Let \( \theta \) be the probability assessed by an uninformed investor that a firm offering \( n_H \) shares at a price \( p_H \) per share is a type bad firm.

\[
\theta = \Pr\{q = B \mid p = p_H, n = n_H\} = \frac{\Pr\{(q = B) \cap (p = p_H, n = n_H)\}}{\Pr\{p = p_H, n = n_H\}} = \frac{\beta(1 - \phi)}{\phi + \beta(1 - \phi)} = 1 - \frac{\phi}{\phi + \beta(1 - \phi)}
\]

When computing \( \theta \), notice that \( \beta(1 - \phi) \) is the proportion of bad firms that pool with the type good firms and \( \phi \) is the proportion of good firms. It follows that \( \theta < 1 - \phi \), i.e., the proportion of bad firms among those firms choosing to offer \( n_H \) shares at price \( p_H \) is lower than the global proportion of bad firms. This occurs because all good firms offer price \( p_H \), but only a proportion of bad firms choose to pool.

Next, we characterize a sufficient condition for investors to produce information about the public period instead of producing no information. When investors produce information, they avoid bidding for a share in a bad firm. Let us measure this expected benefit and compare it to the evaluation cost \( c_p \) in order to find the sufficient condition referred above.

Without information production, the investor does not distinguish between the good and bad firms and pays \( p_H \) to buy one share. The type bad firm only sells \( zn_H \) shares. Thus, the investor’s share in a bad firm is

\[
\left(\frac{1}{m + zn_H}\right).
\]
In the public offer the bad firm only obtains $zI$, which means that the bad firm’s value at time $t = 1$ is $k_B[D + zI]$. The uninformed investor loses because a high price is paid for a fraction

$$\frac{1}{m + zn_H}k_B[D + zI]$$

of the value of a bad firm. Since the value paid is $p_H$, the lost incurred is $p_H - \frac{1}{m + zn_H}k_B(D + zI)$.

The uninformed investor believes that a proportion $\theta$ of the firms offering $p_H \in n_H$ are bad. With this belief, the investor knows that producing public period information, it is possible to identify a proportion $\theta(1 - y)$ of bad firms. When the investor identifies a bad firm, no investment is made, and the loss is prevented. Thus, the expected gain with information production about the public period is

$$\theta(1 - y) \left[ p_H - \frac{1}{m + zn_H}(k_B[D + zI]) \right]$$

Therefore the condition for the investors to produce information about the public period instead of producing no information is that

$$c_p \leq \theta(1 - y) \left[ p_H - \frac{k_B(D + zI)}{m + zn_H} \right].$$

In other words, this inequality states that the investor only produces public period information if the expected gain resulting from the better capacity of identifying bad firms, allowed by this information, is higher than the additional cost of producing full information.
Analogously, the investor choose to produce full information instead of producing no information when
\[ c_f \leq \theta(1 - x) \left[ p_H - \frac{k_B (D + zI)}{m + zn_H} \right]. \]

Given the investor beliefs, the net expected gain with full information is:
\[ \theta(1 - x) \left[ p_H - \frac{k_B (D + zI)}{m + zn_H} \right] - c_f. \]

Similarly, the net expected gain with public information is
\[ \theta(1 - y) \left[ p_H - \frac{k_B (D + zI)}{m + zn_H} \right] - c_p. \]

An uninformed investor chooses to produce full information as opposed to produce public period information if the net expected gain with full information is higher then the net expected gain with public information, which yields:
\[ \theta(y - x) \left[ p_H - \frac{k_B (D + zI)}{m + zn_H} \right] \geq c_f - c_p. \] (18)

Next, we characterize the fraction \( \alpha \) of investors that produce full information in equilibrium. Let us consider two extreme cases. First, consider the case where a large proportion of investors in the Reverse LBO produces information about the public period. The bad firms will then have a high incentive to pool with good firms, because the information produced is not of good quality. In this situation, the investors have incentive to produce more
full information \((\alpha \text{ increases})\). Therefore, this situation is not an equilibrium.

Second, consider the situation where the large majority of investors produce full information. In such case bad firm will have a high cost of pooling with the good firms, implying that bad firms rarely choose the pooling alternative. This creates an incentive for the investors to use more public period information \((\alpha \text{ decreases})\). Hence, we are not in an equilibrium once again. The equilibrium value of \(\alpha\) must be such that bad firms are indifferent between selling \(zn_H\) shares at price \(p_H\) and selling \(n_L\) shares at price \(p_L\). This leads to

\[
\frac{m}{m + n_L} V_B = \frac{m}{m + zn_H} k_B (D + zI),
\]  

(19)

where both \(z\) and \(n_H\) depend on \(\alpha\), and \(V_B\) is given in (2). This equilibrium condition is explained as follows. When bad firms choose to pool they only sell \(zn_H\) shares. Thus, they can only make a fraction \(z\) of the full investment, and the expected return at \(t = 1\) is only \(k_B (D + zI)\). We have

\[
\frac{m}{m + zn_H}
\]

as the fraction of shares belonging to the firm after going public. If bad firms offer \(n_L\) shares, then the full amount to invest \(I\) is obtained. Notice that bad firms know their own type, and there is no information-based uncertainty.

Summarizing, bad firms choosing to pool with good firms face a tradeoff. The advantage is that the share price is higher and the number of shares
that must be sold is lower than in the separating equilibrium. Thus the entrepreneur maintains a higher stake of the firm. The disadvantage is that when the type bad firm pools, only sells $zn_H$ shares. This enables the firm to make only a fraction $z$ of the full investment. The equilibrium value of $\alpha$ reflects the solution of this tradeoff.

Also, there is a relation between the equilibrium values of $\beta$ and $\alpha$. If $\beta$ is close to 1 (bad firm mimic good firms almost for sure), then the benefit from producing full information is high, creating an incentive for a large fraction of investors to produce full information ($\alpha$ high). This imposes a high cost on bad firms that pool with good firms, and this makes $\beta$ to decrease. Thus, $\beta$ close to 1 is *not an equilibrium*. If $\beta$ is close to 0 (bad firms almost never mimic good firms), then the benefit from producing full information is low. Thus, the proportion of investors producing full information is small. This implies that there is a low cost of pooling, and $\beta$ increases. Consequently $\beta$ close to 0 is *not an equilibrium*. This implies that the probability $\beta$ with which bad firms set the high price $p_H$ is determined such that each investor is indifferent between producing information about the public period and producing full information. From (18) this reads

$$\theta(y - x) \left[ p_H - \frac{1}{m + zn_H} k_B (D + zI) \right] = c_f - c_p.$$  

(20)

In order to maximize the entrepreneur’s expected gain, the values for $p_H$ and $n_H$ are established such that the investors’ expected gain when producing
information is 0. Thus, we need equilibrium to satisfy the condition (20), and an additional condition that guarantees that the expected gain from producing information about the public period is 0.

Let us see how much an uninformed investor expects to gain (a priori) if public period information is used. When the investor believes that the firm is type good, with probability \((1 - \theta)\), the expected gain with public period information is\(^{11}\)

\[ (1 - \theta) \left( \frac{1}{m + n_H} V_G - p_H \right), \]

where \(V_G\) is defined in (1). When the investor believes that the firm is bad, we have two possible situations. With probability \(y\) the evaluation is good. Then the investor has a loss, because a high price is paid to buy shares of a bad firm, the expected loss being

\[ y \left[ \frac{1}{m + z n_H} k_B (D + z I) - p_H \right]. \]

With probability \((1 - y)\) the evaluation is bad, and the investor does not invest in the firm. An investment in the zero return risk-free asset is preferred. In all the three situations above the investor pays the evaluation cost \(c_p\). For the expected gain of the investor with public period information to be 0, we

\(^{11}\)Notice that the firm evaluation is always good.
must have

\[(1 - \theta) \left( \frac{1}{m + n_H} V_G - p_H \right) + \theta y \left[ \frac{1}{m + zn_H} k_B (D + zI) - p_H \right] - c_p = 0 \]

or

\[(1 - \theta) \frac{V_G}{m + n_H} + \theta y \frac{k_B (D + zI)}{m + zn_H} - [1 - (1 - y)\theta] p_H - c_p = 0 \quad (21)\]

In equilibrium, the uninformed investor must have an expected gain lower than the obtained by the investor with public period information or with full information. This means that the uninformed investor must have an expected profit lower than or equal to 0. This must be so, so that investors do not have incentive to invest without information, deviating from the equilibrium path. Thus, we must have

\[p_H \geq (1 - \theta) \frac{V_G}{m + n_H} + \theta \frac{k_B (D + zI)}{m + zn_H}.\]

This means that the price to be paid by an uninformed investor must be higher than or equal to his expected gain. Without information, the investor believes that a proportion \((1 - \theta)\) of the firms offering the price \(p_H\) are good, and thus expects to gain \(\frac{1}{m + n_H} V_G\). Moreover, such investor believes that the remaining fraction of firms are bad, and thus the investor expects to gain \(\frac{1}{m + zn_H} k_B [D + zI].\)
The price $p_H$ must be such that enables the good firms to get the amount they need to finance the new project:

$$p_H n_H = I.$$  \hspace{1cm} (22)

Similarly, in order to maximize their expected gain, entrepreneurs will establish $p_L$ and $n_L$ in such a way that an investor is indifferent between bidding and not bidding for shares. In other words, when the firm offers a price $p_L$ it is identified as a bad firm with probability 1

$$p_L = \frac{1}{m + n_L} V_B.$$  \hspace{1cm} (23)

The price $p_L$ and the number of shares $n_L$ must also be such that bad firms can get the entire amount to invest:

$$p_L n_L = I$$  \hspace{1cm} (24)

Concluding the analysis of the above proposition, we have 8 equations reflecting the restrictions characterizing the equilibrium values for the bidding prices and number of shares. These are equations (16,17,19,20,21,22,23,) and (24). Also, we have 8 unknown variables ($p_H^*; p_L^*; n_H^*; n_L^*; z^*; \alpha^*; \beta^*; \theta^*$). The system with this 8 equations defines the values of the variables in the equilibrium in Proposition 3.

Our next result characterizes what happens to the equilibrium price $p_H^*$ when some of the firm’s characteristics change.
Proposition 4 (Comparative Statics). The equilibrium pooling price $p^*_H$ is (a) decreasing in the outsiders’ cost of full information production (i.e., $\frac{\partial p^*_H}{\partial c_f} < 0$), (b) increasing in the outsiders’ cost of production information about the public period (i.e., $\frac{\partial p^*_H}{\partial c_p} > 0$) (c) increasing in the amount to be invested in the new project (i.e., $\frac{\partial p^*_H}{\partial I} > 0$), (d) increasing in the productivity of the firm (i.e., $\frac{\partial p^*_H}{\partial k_G} > 0$). The impact of an increase in investment in the old firm’s projects on the bidding price $p^*_H$ is ambiguous.

The intuition for these results is as follows. As the cost of the full information increases, there are less investors using full information and more using private information because it is relatively cheaper ($\alpha^* \downarrow$) to use private information. This imposes a lower cost in the bad firms that choose to pool, because now the investors have information with lower quality, and therefore identify with less frequency the bad firms that pool. Thus, the number of bad firms pooling increases ($\beta^* \uparrow$). In consequence, the equilibrium probability assessment of uninformed investors that a firm setting a share price $p_H$ is a type bad firm increases ($\uparrow \theta^*$). This implies that investors are willing to pay a lower price for the share ($p^*_H \downarrow$). In this situation, it also follows that, in order to raise the initially fixed amount of funds, the firm must issue more shares ($n^*_H \uparrow$).

If the cost of information about the public period increases, then there are less investors using information about the public period, and there are
more investors using full information, because the relative benefit of the full information increases. This imposes a higher cost in the firms that are pooling, and their number decreases. This implies that the investor will attribute a lower probability that the firm that sells the share at a higher price is a bad firm, and hence the investors are willing to pay a higher price for the share. Consequently, the firm needs to issue less shares to raise the necessary amount to invest.

An increase in the investment in the new project has several joint effects. First, there is an increase in the investor’s advantage in producing full information. This occurs because an increase in the investment increases the loss resulting from investing in a bad firm. Second, there is an increase in the investor’s gain. In fact, an increase in the investment in the new project increases the expected firm’s cash flow at time $t = 1$. However, in spite of this positive effect, the increase in the amount to be invested also implies that the investors must spend a larger amount of funds to have access to the cash flow generated by the old projects. This has a negative impact on the investor’s gain. Yet, it may be seen in the proof of this Proposition that such negative effect is dominated by the positive effect. When the first and second effects are considered together, they imply that the investors have more advantage in producing full information. Consequently, the proportion of bad firms pooling decrease, decreasing also $\theta^*$ and increasing $p_H^*$. 

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Another way to interpret the impact of an increase in $I$, is to notice that the loss of funds for bad firms resulting from pooling also increases. Therefore, the loss in terms of net present value resulting from the pooling is larger. This reduces the proportion of bad firms that chooses to pool.

An increase in the productivity of good firms, increase the value of these firms. This implies an increase in the investor’s gains and an increase in the advantage of producing full information. This reduces the number of bad firms pooling, reduces the investor assessment that the firms are of bad type, increases the shares’ price and decreases the number of shares to be sold.

Finally, if the amount invested in the old projects had been larger, then two contradictory effects would appear. First there would be an increase in the firm’s value, and thus an increase in the investors’ gain. This would induce a higher production of full information. Second, there would be a reduction in the cost of investing in a bad firm, because this type of firm would also become more valuable without additional investment. This implies a reduction in the advantage of producing full information. In conclusion, these two effects make the impact of $D$ in the proportion of firms producing full information ambiguous. This implies that the impact of $D$ in $p^*_H$ is also ambiguous.
3.3 Private Equity vs. Going Public

We have characterized the equilibria when the firms choose the private equity financing and when the firms choose to go public. In this Section we are going to describe the circumstances that make the firms to choose one or another source of financing.

Our main result reflects the following. Given the risk-aversion coefficient \( \rho \) of the VC and the full information cost \( c_f \), there exists a critical value of the public period information such that if \( c_p \) is above that critical value, the entrepreneur prefers to go public; it \( c_p \) is below that critical value, the entrepreneur prefers to be financed by the VC. The critical value of \( c_p \) is shown to be a monotonic function increasing in \( c_f \).

Alternatively, let \( \rho_G(c_p, c_f) \) denote the risk-aversion coefficient of the VC that makes the entrepreneur indifferent between being financed by the VC or the public investors. Such risk-aversion coefficient depends on the structure of the information costs.

Then, the following holds.

**Proposition 5 (Overall equilibrium)** There exists \( \rho_N > 0 \) and \( c_f^U > 0 \) such that for \( k_G > k_G^* \) and \( c_f < c_f^U \), for any given (a) structure of information costs \( (c_p, c_f) \); (b) risk-aversion coefficient \( \rho < \rho_N \) of the VC; (c) restrictions (3) and (15), the entrepreneurs’s choice is characterized as follows. If \( \rho <
Figure 1:

$\rho_G(c_p, c_f)$ the entrepreneur prefers to be financed by the VC; if $\rho > \rho_G(c_p, c_f)$ the entrepreneur prefers to go public. The factor $k_G$ is defined in Proposition 3.\(^{12}\)\(^{13}\)\(^{14}\).

This Proposition has a clear graphical interpretation below. With the help of figure 1, we now discuss some relevant implications of our result.

\(^{12}\)In certain circumstances defined in the appendix, there is a restriction in the going public equilibrium imposing that $c_f$ is not much higher than $c_p$, i.e. $c_p + d > c_f$.

\(^{13}\)The VC's risk-aversion cannot be very large for two reasons. First, to avoid situations where the VC is so risk-averse that demands a remuneration higher than the firm's value (proposition 1). Second, the limitation on the VC risk-aversion is to ensure that an increase in the full evaluation costs it harms more the public financing than the VC financing (proposition 5). This occurs because in the public market, the increase in the evaluation costs is amplified by the large set of investors that evaluate the firm. Moreover, the full information cost can not be very close to the cost of information about the public period to ensure that the investors, in the public equilibrium, do not have incentive in deviate to contracts without information (proposition 5). Finally, the type good firm productivity must be sufficiently higher to guarantee that when the price $p_H$ decrease, the investor that utilizes full information increase its gain (proposition 5).

\(^{14}\)Out-of-equilibrium beliefs that support the equilibrium: if the investors or the VC observe a firm with $\rho > \rho_G(c_p, c_f)$ utilizing the private equity financing, they infer with probability 1 that it is a bad firm. If the investors or the VC observe a firm with $\rho < \rho_G(c_p, c_f)$ going public, they believe that the firm is bad with probability 1. This guarantees that there are not rewarding deviations from equilibrium.
For a given \( \rho \), the curve \( \rho_G(c_p, c_f) = \rho \) is the set of ordered pairs \((c_f, c_p)\) that makes a good entrepreneur indifferent between financing the project using the public market and using private equity. Alternatively, can be seen as the critical value of the public period information defined above, as a function of \( c_f \). This is an increasing function. In fact, when the public period information cost increases (and \( c_f \) is constant), the market becomes a better financing alternative. This occurs because an increase in \( c_p \) increases the share price, given that provides incentives to the investors to use more full information. With a higher share price, the firms need to sell less shares, and the equity’s share to the investors is reduced. To restore the indifference between the two financing alternatives, the full evaluation cost must increase. In fact, an increase in \( c_f \), harms more the public financing than the private equity financing, because in the market there are many investors incurring in evaluation costs. Notice that, above \( c_f^U \) there is no equilibrium.

With both evaluation costs very small, any firm chooses to go public. When the evaluation costs are small and close one to another (for example point \( A \) in figure 1), the investor chooses to go public. This occurs because the proximity between the two costs implies that there is advantage in using full information, which is more precise than the public period information and costs only a little more. Therefore a great proportion of investors will use full information, implying that only a small number of bad firms will choose
to pool. This leads the investors to pay a high price for the shares, and thus the entrepreneur needs to sell only a small number of shares, meaning that the fraction to be given to the investors is small. In consequence, there is advantage in using the market for financing the project.

Let us see what happens when $c_f$ increases (with $c_p$ constant). Recall that the problem of the public financing is that a large number of investors must make the evaluation. In the case of private equity financing, however, only the VC incur in evaluation costs. This implies that the VC becomes steadily a better alternative as the evaluation costs increase. If $c_f$ increases to a value relatively higher than $c_p$ (for example point $B$), the advantage in using full information decreases because, although this information is more accurate, it is relatively more costly. So, the number of investors using full information decrease, inducing a higher number of bad firms to pool with good firms. This reduce the average quality of the firms that offer the high price, meaning that the investor are willing to pay a lower price for the shares. This compels the entrepreneur to offer a higher number of shares, and thus to sell a higher fraction of the firm. The equity’s fraction to offer the VC also increases, because the VC has to be compensated for supporting higher evaluation costs. However, this increase is comparatively lower than the increase in the fraction of equity in the public market. This makes the entrepreneur to prefer the VC.
Let us see what happens if $c_p$ increases (with $c_f$ constant) - suppose, for example, that we pass from point $B$ to point $C$. The increase in $c_p$ has two effects. First, an increase in $c_p$, increases the total evaluation costs, which makes the VC financing more advantageous to the entrepreneur. Second, because $c_f$ and $c_p$ are closer, the relative benefit of full information increases. This induces the investors to use more full information, the number of bad firm’s pooling to decrease, the price of shares to increase and, finally, the number of shares sold to decrease. This last effect makes the going public financing more attractive than the VC financing. However, the second effect dominates the first effect, and in point $C$ the entrepreneur prefers to go public.

There are other situations where the first effect dominates the second effect. When we pass from point $D$ to point $E$, the two described effects
occur, but the evaluation costs in point $E$ are so high that the first effect dominates the second effect, and the firms continues to choose the VC. In point $E$ the two costs are close, and the investors use mainly full information, but the cost of full information is so high that the fraction to be given to the investors is larger than the fraction to be given to the VC. In fact, the loss coming from multiple evaluations by the investors is very high. It then follows that there are firms that never reverse the LBO, because their cost of public and full information is very high.

Notice that, when both $c_p$ and $c_f$ increase, the going public area shrinks. This occurs because the increase in the evaluation costs, increases the loss originated by the multiplication of the evaluation costs in the going public financing. Therefore the firms that have already high evaluation costs (for example firms belonging to industries more difficult to evaluate), tend less to reverse the LBO. In other words, these firms need to be more careful providing information to the investors during the private period, so that their full evaluation costs do not increase to much.

The evaluation costs depend on the amount of information available in the public domain about the firm. Hence, the more information the firm discloses to the public during its private period, the lower will be the full information cost, and the larger will be its propensity to reverse the LBO. On other hand, as the number of years that the firm remains private increases, more
information about the renewed firms comes to public, and the full evaluation cost becomes lower. This means that as the number of years that the firm remains private increases, the propensity to reverse the LBO increases.

As the number of years that a firm remains private increases, the cost of public period information increases. This occurs because as the LBO firm becomes older, the difficulty of accessing the public period information increases and some of this information may get lost. So, for an intermediate value of the cost of full evaluation\(^{15}\), the increase in the cost of public period information increases the propensity to reverse the LBO.

As the time that the firm remains private increase, the public period information’s precision decreases. In fact, as the LBO is more distant in time, the firm becomes steadily more different from the firm that was once public. Then, it follows that the information about the public period becomes less precise in identifying the firm’s quality. This implies an increase in the going public area.

Nevertheless, an increase in the cost of information about the public period constitutes a pressure to an increase in the full information cost (that includes the cost of information about the public period). This does not mean, necessarily, that the full information cost increases. In fact, as the

\(^{15}\)If the firm is in a point like \(B\), in figure (2), an increase in \(c_p\) increases the propensity to reverse the LBO. However, if the firm is in a point like \(D\), in the same figure, then an increase in \(c_p\) does not increase the propensity for the firm to reverse the LBO.
firm remains private for more years, more information about the firm’s private period is released to the public, which is a pressure to the reduction of the full information cost. If the situation is such that, as time passes, the cost of full information increases, this reduces the likelihood that a firm remaining more years private will reverse the LBO. On the contrary, if the cost of full information decreases as time passes, then this increases the likelihood that a firm that remains more years private will go public.

If the firm diversifies its business to industries that are more difficult to evaluate during the LBO, then $c_f$ will be significantly higher than $c_p$, and there will be a higher tendency not to reverse the LBO.

An increase in VC’s risk-aversion increases the firm fraction demanded by the VC. At the same cost of evaluation, this makes the private equity less interesting than the public financing. So, the firms go public for higher evaluation costs (cost of full information and cost of public period information).

We now present what happens to the Reverse LBO decision when the parameters of the model change.

**Corollary 6 (Increase in the investment in old and in new projects)**

An increase in the investment in either the old projects or in the new project has an ambiguous impact on the decision of reverting the LBO.

First, an increase in the investment in the new project has an ambiguous impact on the fraction to be given to the VC, as follow from Proposition 2.
Similarly, the impact of an increase in $I$ in the equity fraction to investors is also uncertain. In fact, an increase in $I$ has two effects of opposite signals on the equity fraction to investors. First, the value of the firm per unit of investment is reduced. This makes the investors to demand a higher fraction of the firm’s value. Second, an increase in $I$ enables the firm to sell a lower fraction of equity to investors. Altogether, an increase in the new project’s investment has a non-defined effect on the decision of reverting the LBO.

In the same way, an increase in the amount invested in old projects has an ambiguous effect on the fraction of equity to investors, because we do not know, in first place, the impact on $p^*_H$. Due to this, the impact of $D$ in the decision of going public is uncertain. However, if an increase in $D$ increases $p^*_H$, this would decrease the fraction of equity to investors. This would imply that an increase in $D$ induces the firms to go public for higher costs (provided that the VC is not too risk-averse).

**Corollary 7 (Increase in firm’s risk)** As the investment risk (either $\sigma^2_{\epsilon D}$ and/or $\sigma^2_{\epsilon I}$) or the covariance between the cash flows of the old and new projects $\sigma_{DI}$ increases, the upper bounds in the evaluation costs preventing firms to go public increase.

When faced with firms with higher investment uncertainty (for example, resulting from higher technological uncertainty), the risk-averse VC demand
a higher risk premium. This makes the VC less attractive in relation to the public financing alternative. With the same evaluation costs, the attractiveness of the public financing increases. So, good firms prefer to go public for higher evaluation costs. In order to maintain the pooling, bad firms follow the good firms, and also go public for higher costs. So, firms with higher investment uncertainty go public for higher evaluation costs.

A higher covariance between the cash flows of the old and new projects of a firm implies in a higher risk of the cash flows. This means that firms involved in a strategy of diversification revert the LBO for higher evaluation costs.

**Corollary 8 (Increase in productivity)** If there is a productivity shock in an industry such that \( k_G \) increases, then there exists \( \rho_{M3} > 0 \) such that, for \( \rho < \rho_{M3} \), the upper bounds in the evaluation costs preventing firms to go public in that industry increase.

An increase in the productivity of good firms increases the value generated by them. Hence, the VC demands a lower fraction of the firm, as established in Proposition 1. On the public financing side, the advantage of producing information of higher quality, in order to identify the type good firm, increases. Thus, the number of investors producing full information increase, and the firm fraction to the investors decreases. Taking into ac-
count these two effects and the fact that the VC is not too risk-averse\textsuperscript{16}, an increase in the firm productivity reduces more the fraction to the investors then the fraction to the VC. This makes the going public alternative more appealing. In conclusion, with higher productivity, the firms go public for higher evaluation costs.

The last result can be interpreted from two different points of view. First, we may conclude that firms with high productivity go public for evaluation costs' levels that would not allow less productive firms to go public. Second, this result indicates that, if there exists a productivity shock that affect all firms in an industry, then a large number of firms may reverse the LBO\textsuperscript{17}.

4 Appendix

4.1 Proof of Proposition 1

We have a pooling equilibrium where both types of firms offer the same contract \( \{ s_{g f}^*, s_{bf}^* \} \), that is contingent in the result of the evaluation, and the VC performs the evaluation.

First, we will determine the equity share that each firm will offer to

\textsuperscript{16}We assume that the VC risk-aversion is not too high. An increase in \( I \), increase the VC utility. This allows the entrepreneur to decrease the firm share to the VC. When the VC risk-aversion increases, his utility sensitivity to a reduction in the firm share decreases, because he is now more worried with the risk. Hence, when the investment in the firm increases, the entrepreneur can reduce more the VC’s share when the VC risk-aversion is higher.

\textsuperscript{17}Let us assume that firms are uniformly distributed in the plan \((c_p, c_f)\). If \( kG \) increases in the industry, the firms located in the area that becomes a going public area go all public simultaneously. This reasoning is similar to the one presented in Chemmanur and Fulghieri (1999).
the VC. The firm will offer to the VC the lowest possible equity share that ensures its participation. **If the evaluation is bad**, then the entrepreneur is by sure of type bad. The equity fraction to be offered to the VC is obtained by taking the smallest \( s_{b_f}^* (c_f, \rho) \) that satisfy \( (??) b \). To ensure that \( s_{b_f}^* < 1 \) and that the VC is willing to finance the entrepreneur\(^{18} \), we assume that \( \rho < \rho_1 \), where 

\[
\rho_1 = \frac{V_{b_f} - I - c_f}{\sigma_{2f}^2 + \sigma_{2f}^2 + 2 \sigma_{Df}}.
\]

The parameter \( \rho_1 \) is obtained taking \( s_{b_f} = 1 \) in equation \( (??) b \) and resolving the equation in order to \( \rho \). If \( \rho > \rho_1 \), then this will imply that \( s_{b_f} > 1 \). This means that the VC is so risk averse that demands an equity share greater then 1. So, we put \( \rho < \rho_1 \) in order to ensure that \( s_{b_f} < 1 \). Can be verified, by implicit differentiation that \( \frac{\partial s_{b_f}^*}{\partial c_f} > 0 \) (see proposition 2).

**If the evaluation outcome is good**, then the entrepreneur will offer the equity share \( s_{g_f}^* (c_f, \rho) \), which is the smallest root in equation \( (??) a \). We assume that 

\[
\rho < \rho_2 = \frac{V_{g_f} - I - c_f}{\sigma_{2f}^2 + \sigma_{2f}^2 + 2 \sigma_{Df}}.
\]

where \( \rho_2 \) is obtained taking \( s_{g_f} = 1 \) in equation \( (??) a \) and resolving the equation in order to \( \rho \). Note that \( \frac{\partial s_{g_f}^*}{\partial c_f} > 0 \) (see proposition 2).

Given the out-of-equilibrium beliefs, a firm \((Good \ or \ Bad)\) may **deviate from the pooling contract** and identify himself as a type bad firm. These firm has two choices: offer an unconditional price contract, which does not require information production, or offer a contract with information about

\(^{18} \text{If } s_{b_f}^* > 1, \text{then the VC is not willing to finance the entrepreneur, because the share of equity that he demand is higher then all equity of the firm.} \)
the public period.

If the entrepreneur offers the unconditional price contract, the share of equity to give the VC is \( s^*_B(\rho) \), that fulfill the equation (??). On the other hand, if the firm offers a contract with information about the public period, he is also identified as bad, but in this case the VC has to incur in evaluation costs. So, the equity share that the VC demands is higher than in the last case, and satisfy the equation: \( s_{bf}V_{bf} - \rho s_{bf}^2 (\sigma^2_{\xi D} + \sigma^2_{\xi I} + 2\sigma_{DI}) = I + c_f \) (see (??)). In conclusion, the best deviating contract is the unconditional price contract.

Second, we will proof the existence of equilibrium. We must guarantee that each of the entrepreneurs prefer the pooling contract to the best desviating contract. The type good firm will be always identified as good, and so will give the VC \( s^*_g \).

Hence, the type good firm will prefer the pooling contract to the unconditional price contract if \( s^*_g(c_f, \rho) < s^*_B(\rho) \). We have, \( s^*_g(0, 0) = \frac{I}{V_{gf}} \) and \( s^*_B(0) = \frac{I}{V_B} \) (see figure (3)). This means that \( s^*_g(0, 0) < s^*_B(0) \), because \( V_{gf} > V_B \). Let \( \rho_3 \) be minimum \( \rho \) such that \( s^*_g(0, \rho) = s^*_B(\rho) \) (if there is no such a \( \rho \), set \( \rho_3 = \infty \)). With \( \rho < \rho_3 \), there exist a \( c_{fg} \) such that \( s^*_g(c_{fg}, \rho) = s^*_B(\rho) \). When we pass from \( \rho_3 \) to \( \rho \), we have \( s^*_g(0, \rho) < s^*_B(\rho) \). Thus, \( c \) must growth until \( \rho_3 \) in order to have \( s^*_g = s^*_B \). With \( c < c_{fg} \), we have \( s^*_g(c_f, \rho) < s^*_B(\rho) \). In conclusion, with \( c_f < c_{fg}(\rho) \) and \( \rho < \rho_3 \), the
type good entrepreneur prefers the pooling contract, with \( \{s_g^*, s_b^*\} \), than the deviating contract, with \( s_B^* \).

Now, let us proof that the type bad firm prefers the pooling contract to the unconditional price contract. With the pooling contract, the equity share that the type bad firm expect to give to the VC is

\[
s_f^*(c_f, \rho) = x s_g^*(c_f, \rho) + (1 - x) s_b^*(c_f, \rho)
\]  

This type of entrepreneur prefers the pooling contract if \( s_f^*(c, \rho) < s_B^*(\rho) \).

We have

\[
s_f^*(0, 0) = x s_g^*(0, 0) + (1 - x) s_b^*(0, 0) = x \frac{f}{V_g} + (1 - x) \frac{f}{V_b}.
\]

We know that \( V_g > V_B = V_b \), thus we have \( s_f^*(0, 0) < s_B^*(0) = \frac{f}{V_B} \). Let \( \rho_4 \) be the minimum \( \rho \) for which \( s_f^*(0, \rho) = s_B^*(\rho) \) (if there is no such a \( \rho \) set \( \rho_4 = \infty \)). With \( \rho < \rho_4 \) we have \( s_f^*(0, \rho) < s_B^*(\rho) \). So, because \( s_f^*(c_f, \rho) \) is increasing in \( c_f \),
there exist a $\{B(p)\}$ such that $s^*_f(c_b, \rho) = s^*_B(\rho)$. In conclusion, with $c_f < c_{f_b}$ (\rho) and $\rho < \rho_4$, the type bad firm prefers the pooling contract.

The proof is completed with $\rho_v = \min \{\rho_1, \rho_2, \rho_3, \rho_4\}$ and $c_{f_v}(\rho) = \min \{c_{f_b}(\rho), c_{f_b}(\rho)\}$.

### 4.2 Proof of Proposition 2

$s^*_{gf}$ is defined implicitly by (?? a): $F = s_{gf}V_{gf} - \rho s^2_{gf}(\sigma^2_{gf} + \sigma^2_{\epsilon D} + \sigma^2_{\epsilon I} + 2\sigma_{DI}) - I - c_f = 0 \iff s^*_{gf} = f(I, c_f, \rho, V_{gf}, \sigma^2_{gf}, \sigma^2_{\epsilon D}, \sigma^2_{\epsilon I}, 2\sigma_{DI})$. $s^*_{gf}$ is given implicitly by the following equation (?? b): $T = s_{bf}V_{bf} - \rho s^2_{bf}(\sigma^2_{\epsilon D} + \sigma^2_{\epsilon I} + 2\sigma_{DI}) - I - c_f = 0 \iff s^*_{bf} = t(I, c_f, \rho, V_{bf}, \sigma^2_{\epsilon D}, \sigma^2_{\epsilon I}, 2\sigma_{DI})$. Doing implicit differentiation, we can proof proposition 2.

We will proof that an increase in $c_f$ increases $s^*_{gf}$ and $s^*_{bf}$. An increase in the VC equity share is equivalent to an increase in the VC wealth. In order to the utility function have interesting economic behaviour, the agent utility must grow with his wealth (Mascollel, p. 185). Thus is reasonable to assume that $\frac{\partial F}{\partial s^*_{gf}} > 0$ and $\frac{\partial T}{\partial s^*_{bf}} > 0$. Thus we have $\frac{\partial s^*_{gf}}{\partial c_f} = -\frac{\partial F}{\partial s_{gf}} = -\frac{1}{\frac{\partial F}{\partial s_{gf}}} > 0$.

Similarly $\frac{\partial s^*_{bf}}{\partial c_f} = -\frac{\partial T}{\partial s_{bf}} = -\frac{1}{\frac{\partial T}{\partial s_{bf}}} > 0$.

An increase in $\rho$ increases $s^*_{gf}$ and $s^*_{bf}$, $\frac{\partial s^*_{bf}}{\partial \rho} = -\frac{\partial T}{\partial \rho} = \frac{s^2_{bf}(\sigma^2_{\epsilon D} + \sigma^2_{\epsilon I} + 2\sigma_{DI})}{\frac{\partial T}{\partial s_{bf}}} > 0$ and $\frac{\partial s^*_{gf}}{\partial \rho} = -\frac{\partial F}{\partial \rho} = \frac{s^2_{gf}(\sigma^2_{gf} + \sigma^2_{\epsilon D} + \sigma^2_{\epsilon I} + 2\sigma_{DI})}{\frac{\partial F}{\partial s_{gf}}} > 0$.

An increase in $\sigma^2_{\epsilon D}$ and $\sigma^2_{\epsilon I}$ increases $s^*_{gf}$ and $s^*_{bf}$, $\frac{\partial s^*_{bf}}{\partial \sigma_{\epsilon D}} = \frac{\partial s^*_{bf}}{\partial \sigma_{\epsilon I}} = \frac{\rho s^2_{bf}}{\frac{\partial s^*_{bf}}{\partial s_{bf}}} > 0$ and $\frac{\partial s^*_{gf}}{\partial \sigma_{\epsilon D}} = \frac{\partial s^*_{gf}}{\partial \sigma_{\epsilon I}} = \frac{\rho s^2_{gf}}{\frac{\partial s^*_{gf}}{\partial s_{gf}}} > 0$.
An increase in $\sigma_{Df}$ increases $s_{gf}^*$ and $s_{bf}^*$, 
\[ \frac{\partial s_{gf}^*}{\partial \sigma_{Df}} = -\frac{2\rho s_{bf}^2}{\rho} > 0, \quad \frac{\partial s_{bf}^*}{\partial \sigma_{Df}} = -\frac{2\rho s_{bf}^2}{\rho} > 0. \]

An increase in $I$ has an ambiguous impact on $s_{gf}^*$ and $s_{bf}^*$, 
\[ \frac{\partial s_{bf}^*}{\partial I} = -\frac{\partial \sigma_{bf}^*}{\partial I} \]
and
\[ \frac{\partial s_{bf}^*}{\partial I} = -\frac{\partial \sigma_{bf}^*}{\partial I}. \]
From (?? b) we have
\[ s_{bf}^* k_B \left( \frac{I+1}{I} \right) = \frac{\rho s_{bf}^2 (\sigma_{gf}^2 + \sigma_{bf}^2 + 2\sigma_{Df})}{f} > 0. \]
If the difference between $k_B = \frac{\partial V_{bf}}{\partial I}$ and $k_B \left( \frac{I+1}{I} \right) = \frac{V_{bf}}{f}$ is small (\( \frac{I+1}{I} \) is not too large), then we have the guarantee that $s_{bf}^* k_B - 1 > 0$. 
\[ \frac{\partial s_{gf}^*}{\partial I} = -\frac{\partial \sigma_{gf}^*}{\partial I} = -s_{gf}^* \frac{\partial \sigma_{gf}^*}{\partial I} = -s_{gf}^* \frac{\partial \sigma_{bf}^*}{\partial I}. \]
\[ V_{gf} = k_G (D + I) \text{Prob}(q = G|e_f = g_f) + k_B (D + I) \left[ 1 - \text{Prob}(q = G|e_f = g_f) \right]. \]
This implies that 
\[ \frac{\partial \sigma_{gf}^*}{\partial I} = k_G \text{Prob}(q = G|e_f = g_f) + k_B \left( \frac{I+1}{I} \right) \left[ 1 - \text{Prob}(q = G|e_f = g_f) \right]. \]
\[ \frac{V_{gf}}{I} = k_G \left( \frac{I+1}{I} \right) \frac{\partial \sigma_{gf}^*}{\partial I}. \]
We have
\[ \frac{V_{gf}}{f} \left( \frac{I+1}{I} \right) \frac{\partial \sigma_{gf}^*}{\partial I} > 0. \]
So, if $\frac{\partial \sigma_{gf}^*}{\partial I}$ is not much smaller then $\frac{V_{gf}}{f} \left( \frac{I+1}{I} \right)$ (\( \frac{I+1}{I} \) is not too large), we have the guarantee that 
\[ s_{gf}^* \frac{\partial \sigma_{gf}^*}{\partial I} - 1 > 0. \]
This also implies that $\frac{\partial s_{gf}^*}{\partial I} < 0$. 

An increase in $D$ decreases $s_{gf}^*$ and $s_{bf}^*$, 
\[ \frac{\partial s_{gf}^*}{\partial D} = -\frac{\partial \sigma_{gf}^*}{\partial D} = -s_{gf}^* \frac{\partial \sigma_{gf}^*}{\partial D} < 0. \]

An increase in $k_G$ decreases $s_{gf}^*$ and $s_{bf}^*$, 
\[ \frac{\partial s_{gf}^*}{\partial k_G} = -\frac{\partial \sigma_{gf}^*}{\partial k_G} = -s_{gf}^* \frac{\partial \sigma_{gf}^*}{\partial s_{bf}^*} < 0. \]

An increase in $k_B$ decreases $s_{gf}^*$ and $s_{bf}^*$, 
\[ \frac{\partial s_{gf}^*}{\partial k_B} = -\frac{\partial \sigma_{gf}^*}{\partial k_B} = -s_{gf}^* \frac{\partial \sigma_{gf}^*}{\partial s_{bf}^*} < 0. \]
4.3 Proof of Proposition 3

Beginning with equation (??) and simplifying we get, after some manipulation, \( \frac{1}{m+n_H} = \frac{D+I}{D+zI} \frac{1}{m+n_L} \). From (??) with \( V_G = k_G(D + I) \) and \( V_B = k_B(D + I) \), we have:

\[
(1 - \theta) \left[ \frac{1}{m+n_H} (k_G [D + I]) + \theta y \left[ \frac{1}{m+n_H} (k_B [D + zI]) \right] \right] - [1 - (1 - y)\theta] p_H - c_p = 0.
\]

Dividing by \( I \), we have:

\[
(1 - \theta) \left[ \frac{1}{m+n_H} (k_G \frac{P}{T} + 1) \right] + \theta y \left[ \frac{1}{m+n_H} (k_B \frac{P}{T} + z) \right] = [1 - (1 - y)\theta] \frac{1}{n_H} + \frac{c_p}{\theta}.
\]

From (??) we have that \( p_L = \frac{1}{m+n_L} k_B(D + I) \). From (??) we have \( p_L n_L = \frac{1}{n_L} \), and thus \( \frac{1}{n_L} = \frac{1}{m+n_L} k_B(D + I) \). From this last equation we have \( n_L = \frac{P}{D+I} k_B^{-1} \). So \( \frac{1}{m+n_L} k_B = \frac{I}{D+I} \left( \frac{M+1}{m} k_B^{-1} \right) \). We get finally:

\[
H(\theta, n_H) = (1 - \theta) \frac{k_G n_H}{m+n_H} \left( \frac{P}{T} + 1 \right) + \theta y \frac{(P+1)k_B^{-1}}{m} n_H - [1 - (1 - y)\theta] \frac{1}{n_H} - \frac{c_p}{\theta} = 0
\]

From (??) we have \( \theta(y - x) \left[ p_H - \frac{1}{m+n_H} (k_B [D + zI]) \right] = c_f - c_p \). Dividing by \( I \), we have:

\[
\theta(y - x) \left[ \frac{P}{T} - \frac{k_B}{m+n_H} \frac{D+zI}{I} \right] = \frac{c_f - c_p}{I}.
\]

We know that \( \frac{1}{m+n_H} = \frac{D+I}{D+zI} \frac{1}{m+n_L} \) and \( \frac{P}{T} = \frac{1}{n_H} \), which implies \( \theta(y - x) \left[ \frac{1}{n_H} - \frac{D+I}{I} \frac{k_B}{(m+n_L)} \right] = \frac{c_f - c_p}{I} \). Moreover, \( \frac{k_B}{m+n_L} = \frac{I}{D+I} \left( \frac{P+1}{m} k_B^{-1} \right) \), and so:

\[
G(\theta, n_H, \cdot) = \theta(y - x) \left[ \frac{1}{n_H} - \frac{(P+1)k_B^{-1}}{m} \right] = \frac{c_f - c_p}{I}.
\]

We determine \( \theta^* \) and \( n^*_H \) from the following system of equations:
(a) \( H(\theta, n_H) = (1 - \theta) \frac{k_G n_H}{m + n_H} \left( \frac{D}{I} + 1 \right) + \theta y \left( \frac{D}{I} + 1 \right) - \frac{1}{m} n_H - [1 - (1 - y)\theta] - \frac{c_p}{I} n_H = 0 \) \hspace{1cm} (26)

(b) \( G(\theta, n_H) = \theta (y - x) \left[ \frac{1}{n_H} - \frac{(D+1)k_B-1}{m} \right] - \frac{c_f-c_p}{I} = 0 \)

With \( 0 \leq \theta \leq 1 - \phi \). From (??): \( \theta = 1 - \frac{\phi}{\phi+\beta(1-\phi)} \). We have \( \frac{\phi}{\phi+\beta(1-\phi)} > \phi \), because \( \phi + \beta(1-\phi) < 1 \). So \( 1 - \frac{\phi}{\phi+\beta(1-\phi)} < 1 - \phi \).

Equations (??) and (??) enable to obtain \( r_L^* \) and \( n_L^* \).

The equation (??) may be solved for \( z^* : \frac{1}{m + zn_H} = \frac{D+I}{D+2I} \left( \frac{1}{m + n_L} \right) \leftrightarrow z^* = \frac{I(m-Dn_L^*)}{I(m+n_L^*)-n_H^*(M+I)} \).

The equation (??) may be solved for \( \alpha^* : z^* n_H^* = N [(1 - \alpha)y + \alpha x] \leftrightarrow \alpha^* = \frac{(z^* n_H^* - N y)}{(x-y)N} \).

Note that \( \alpha^* \) must be between 0 and 1. Let us guarantee that \( \alpha^* < 1 \).

\( \alpha^* < 1 \Leftrightarrow \frac{(z^* n_H^* - N y)}{(x-y)N} < 1 \Leftrightarrow x > z^* n_H^* = (1 - \alpha^*) y + \alpha^* x \). Because \( y > x \) and \( 0 < \alpha^* < 1 \), we \( (1 - \alpha^*) y + \alpha^* x > x \). So, \( \alpha^* < 1 \) is always guaranteed.

Equation (??) may be solved for \( \beta^* : \theta^* = 1 - \frac{\phi}{\phi+\beta(1-\phi)} \).

Equation (??) \( (p_H n_H^* = I) \) may be solved for \( p_H^* \).

Below, we will guarantee the existence of a solution of the system (26).

The equation (26 b) can be solved for \( n_H : \theta (y - x) \left[ \frac{1}{n_H} - \frac{(k_B-1)}{m} \right] - \frac{c_f-c_p}{I} = 0 \).
\[ n_H = g(\theta) = \theta \left[ \frac{c_f - c_p}{I(y - x)} + \frac{\theta(k_B - 1)}{m} \right]^{-1} \] (27)

Can be proved that \( g(\theta) \) is increasing in \( \theta \): \( \frac{\partial g(\theta)}{\partial \theta} = \frac{c_f - c_p}{I(y - x)} + \frac{\theta(k_B - 1)}{m} > 0, \)
because \((c_f - c_p) > 0\) and \((y - x) > 0\). Note that \( g(0) = 0 \) and define \( \hat{n}_H = g(1 - \phi) \).

The equation (26 a) can be rewritten as:

\[
\left( \theta y \frac{(k_B - 1) - c_p}{m} - \frac{c_p}{I} \right) n_H^2 + \left[ (1 - \theta)k_G + \theta y(k_B - 1) - (1 - (1 - y)\theta) - \frac{c_p m}{I} \right] n_H
\]
\[-(1 - (1 - y)\theta)m \]

Define \( n_H = h(\theta) \) as the positive root of equation (28):

\[ n_H = h(\theta) \] (29)

The discriminant of equation (28) is

\[
d = \left[ (1 - \theta)k_G + \theta y(k_B - 1) - (1 - (1 - y)\theta) - \frac{c_p m}{I} \right]^2 + 4 \left( \theta y \frac{(k_B - 1) - c_p}{m} \right) \left( (1 - (1 - y)\theta) m \right)
\]

\[
(1 - (1 - y)\theta)m \cdot D > 0 \iff k_G > D = \frac{1}{1 - \theta} \left( 2 \left( \theta y \frac{(k_B - 1) - c_p}{m} \right) \right) \left( (1 - (1 - y)\theta) m \right)^{1/2} - \theta y
\]

Define \( \theta_M = \text{argmax}(D) \), with \( 0 < \theta < 1 - \phi \). With \( k_G > k_G1 = D(\theta_M) \), then

we have \( d > 0 \) for all \( 0 < \theta < 1 - \phi \).

Implicit differentiation of (26 a) gives \( \frac{\partial n_H(\theta)}{\partial \theta} = -\frac{\partial H}{\partial n_H} \). Let us define the

signal of this derivative.
\[
\frac{\partial H}{\partial \theta} = -\frac{k_G}{m+n_H} + y \frac{(k_B-1)}{m} n_H + (1-y) - \frac{n_H}{m+n_H} \left[ \frac{V_G}{m+n_H} - p_H \right] + \frac{y}{n_H} \left[ \frac{(k_B-1)}{m} - \frac{1}{n_H} \right].
\]

\[
\left[ \frac{V_G}{m+n_H} - p_H \right]
\]

is the investor gain when the firm is type good, and so is higher than 0. The investor gain when the firm is bad is

\[
I \left[ \frac{(k_B-1)}{m} - \frac{1}{n_H} \right] < 0.
\]

Then we have

\[
\left[ \frac{(k_B-1)}{m} - \frac{1}{n_H} \right] < 0.
\]

This means that

\[
\frac{\partial H}{\partial \theta} < 0
\]

The proportion of type bad firms increase, the gain of the investor with public information decrease.

\[
\frac{\partial H}{\partial n_H} = (1-\theta) \frac{k_G m}{m+n_H} + \theta y \frac{(k_B-1)}{m} c_p.
\]

This expression doesn’t have a defined signal. In Chemmanur and Fulghieri (1999) this expression was positive, which means that an increase in \( n_H \) (is the same that a decrease in the price), conducts to a increase in the gain of the investor with public information.

Define \( L = \frac{(m+n_H)^2}{m(1-\theta)} \left[ \frac{c_p}{T} - \theta y \frac{(k_B-1)}{m} \right] \). Found \( (n_{H1}, \theta_{M1}) \) that maximize \( L \), with \( 0 < n_H < g(1-\phi) \) and \( 0 < \theta < 1 - \phi \). With \( k_G > k_G2 = L(n_{H1}, \theta_{M1}) \), we always have:

\[
\frac{\partial H}{\partial n_H} > 0
\]

Finally consider \( k_G > k_G = max(k_{G1}, k_G2) \). In conclusion we have \( \frac{\partial n_H(\theta)}{\partial \theta} > 0 \).
Figure 4:

Note that $H(0, n_H) = -c_p n_H^2 + [k_G - 1 - c_p m] n_H - m = 0$. The positive root of this equation is $h(0)$ and exist because $k_G < k_G$. Define $n_H = h(1 - \phi)$.

We have $0 = g(0) < h(0)$, and assuming that $\hat{n}_H > n_H$, and the continuity of equation (27)\textsuperscript{19} ensure the solution of equation (26).

From equation (27), condition $\hat{n}_H > n_H$ holds if and only if $\hat{n}_H = g(1 - \phi) = (1 - \phi) \left[ \frac{c_f - c_p}{I(y - x)} + \frac{(1 - \phi)(k_B - 1)}{m} \right]^{-1} > n_H \iff$

\[ c_f - c_p < d_s = (1 - \phi) (y - x) \left[ \frac{1}{n_H} - \frac{(k_B - 1)}{m} \right] I \]

\textbf{The uninformed investor} has negative expected gain: $p_H \geq (1 - \theta) \frac{1}{m + n_H} V_G + \theta \frac{z}{m + z n_H} V_B$

Remembering equation (??) we have: $(1 - \theta) \left[ \frac{1}{m + n_H} V_G \right] = [1 - (1 - y)\theta] p_H +$

\textsuperscript{19}Não é necessária a continuidade de A3?

\textsuperscript{20}As formas das curvas no gráfico são meramente ilustrativas.
\[ c_p - \theta y \left[ \frac{z}{m+zn_H} V_B \right] \]

Substituting this result in the relation above: 
\[ p_H \geq \left[ 1 - (1 - y)\theta \right] p_H + c_p - \theta y \left[ \frac{z}{m+zn_H} V_B \right] + \theta \frac{z}{m+zn_H} V_B \Rightarrow 0 \geq -(1 - y)\theta p_H + c_p + (1 - y)\theta \left[ \frac{z}{m+zn_H} V_B \right] \]

From equation (29) we have:
\[ p_H = \frac{c_f - c_p}{\theta(y-x)} + \frac{z}{m+zn_H} V_B. \]

Substituting this result in the last relation, we obtain:
\[ 0 \geq -(1 - y)\theta \left[ \frac{c_f - c_p}{\theta(y-x)} + \frac{z}{m+zn_H} V_B \right] + c_p + (1 - y)\theta \left[ \frac{z}{m+zn_H} V_B \right] \]
\[ \Rightarrow \frac{c_f}{\nu} \geq \frac{(1-x)}{(1-y)}. \]

### 4.4 Proof of Proposition 4

From equation (29), we know that: 
\[ p^*_H = \frac{1}{n_H}. \]

So, 
\[ \frac{\partial p^*_H}{\partial c_f} = -\frac{1}{n_H^2}. \]

Next we will determine the signal of \( \frac{\partial n_H^*}{\partial c_f} \).

The system of equations (26) defines two equations: \( n^*_H = j(\theta, ...) \) and \( \theta^* = f(n_H, ...) \). Let us differentiate implicitly the system of equations (26):

\[
\begin{bmatrix}
\frac{\partial j}{\partial c_f} \\
\frac{\partial \theta}{\partial c_f}
\end{bmatrix} = - \begin{bmatrix}
\frac{\partial H}{\partial n_H} & \frac{\partial H}{\partial G} \\
\frac{\partial G}{\partial n_H} & \frac{\partial G}{\partial \theta}
\end{bmatrix}^{-1} \begin{bmatrix}
\frac{\partial H}{\partial c_f} \\
\frac{\partial G}{\partial c_f}
\end{bmatrix}
\]

\[
= - \begin{bmatrix}
\frac{\partial H}{\partial n_H} & \frac{\partial H}{\partial G} \\
\frac{\partial G}{\partial n_H} & \frac{\partial G}{\partial \theta}
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{\partial H}{\partial n_H} & \frac{\partial H}{\partial G} \\
\frac{\partial G}{\partial n_H} & \frac{\partial G}{\partial \theta}
\end{bmatrix} \begin{bmatrix}
(-\frac{1}{2}) \\
\frac{1}{2}
\end{bmatrix}
\]

(32)

So, 
\[ \frac{\partial j}{\partial c_f} = \frac{\partial n_H^*}{\partial c_f} = \left( \begin{bmatrix}
\frac{\partial H}{\partial n_H} \\
\frac{\partial G}{\partial n_H}
\end{bmatrix} \right) \left( -\frac{1}{2} \right) = \frac{\partial H}{\partial n_H} \frac{1}{2}. \]

Let us see what is the signal of \( \frac{\partial H}{\partial \theta} - \frac{\partial H}{\partial n_H} \frac{\partial G}{\partial \theta}. \)

Consider \( h(\theta) \) (29) and \( g(\theta) \) (27). We have:
\[ h(\theta) = \frac{\partial h(\theta)}{\partial \theta} = -\frac{\partial H}{\partial n_H} \text{ and } \]
\[ g(\theta) = \frac{\partial g(\theta)}{\partial \theta} = -\frac{\partial G}{\partial n_H}. \]

Since \( h(\theta) < g(\theta) \) (see figure (4)), we have 
\[ -\frac{\partial H}{\partial n_H} < \frac{\partial G}{\partial n_H} \text{ and } \]
\[ h(\theta) - g(\theta) < 0. \]

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\[-\frac{\partial G}{\partial n_H} \Leftrightarrow \frac{\partial H}{\partial n_H} \frac{\partial G}{\partial n_H} > 0.\] We have \(\frac{\partial H}{\partial n_H} > 0\) (from (31)) and

\[\frac{\partial G}{\partial n_H} = \theta(y - x) \left[ -\frac{1}{n_H^2} \right] < 0 \quad (33)\]

So, \(\frac{\partial H}{\partial n_H} \frac{\partial G}{\partial n_H} < 0\). This implies that we have

\[\frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} - \frac{\partial G}{\partial \theta} \frac{\partial H}{\partial n_H} < 0 \quad (34)\]

With \(\frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} - \frac{\partial G}{\partial \theta} \frac{\partial H}{\partial n_H} < 0\) and \(\frac{\partial H}{\partial \theta} < 0\) (from (30)), then \(\frac{\partial H}{\partial n_H} \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} > 0\).

Finally,

\[\frac{\partial n^*_H}{\partial c_f} = \frac{\partial H}{\partial n_H} \frac{\partial H}{\partial n_H} \frac{\partial G}{\partial \theta} - \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} \mu \left( \frac{1}{I} \right) > 0 \quad (35)\]

which implies that \(\frac{\partial n^*_H}{\partial c_f} = \frac{-I}{\partial n_H^2} < 0\).

Now let us determine the signal of \(\frac{\partial \theta^*_H}{\partial c_f} = \frac{\partial f}{\partial c_f} = \frac{\partial H}{\partial n_H} \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} \mu \left( \frac{1}{I} \right) \) (see (32)).

We know that \(\frac{\partial H}{\partial n_H} \frac{\partial G}{\partial \theta} - \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} > 0\) (from (34)) and that \(\frac{\partial H}{\partial \theta} > 0\) (from (31)). So

\[\frac{\partial \theta^*_H}{\partial c_f} = \frac{\partial H}{\partial n_H} \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} \mu \left( \frac{1}{I} \right) > 0 \quad (36)\]

\[\frac{\partial n^*_H}{\partial c_p} = \frac{\partial}{\partial c_p} \left[ \frac{\partial f}{\partial c_p} \right] = \frac{-I}{\partial n_H^2} \left( \frac{n_H}{n_H^2} \right)^2\]
\[
\begin{bmatrix}
\frac{\partial g}{\partial f} \\
\frac{\partial f}{\partial c_p}
\end{bmatrix}
= -\begin{bmatrix}
\frac{\partial G}{\partial n_H} & \frac{\partial H}{\partial n_H} \\
\frac{\partial G}{\partial c_p} & \frac{\partial H}{\partial c_p}
\end{bmatrix}
\begin{bmatrix}
\frac{1}{T} \\
\frac{1}{T}
\end{bmatrix}
\]

\[
\frac{\partial n^*_H}{\partial c_p} = \frac{\partial n^*_i}{\partial c_p} = \begin{bmatrix}
\frac{1}{T} \\
\frac{1}{T}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial G}{\partial n_H} + \frac{\partial H}{\partial n_H} \\
\frac{\partial G}{\partial c_p} + \frac{\partial H}{\partial c_p}
\end{bmatrix}
\]

We know that \(\frac{\partial H}{\partial n_H} \frac{\partial G}{\partial n_H} - \frac{\partial H}{\partial n_H} \frac{\partial G}{\partial n_H} > 0\) (from (34)) and that \(\frac{\partial H}{\partial c_p} < 0\) (from (30)). From (26 b) we get

\[
\frac{\partial G}{\partial \theta} = (y - x) \left[ \frac{1}{n_H} - \frac{(k_B - 1)}{m} \right] > 0 \tag{38}
\]

So we have \(\frac{\partial G}{\partial \theta} + \frac{\partial H}{\partial \theta} = ((y - x) \left[ \frac{1}{n_H} - \frac{(k_B - 1)}{m} \right] + \left( \frac{k_G n_H}{m + n_H} + y \frac{(k_B - 1)}{m} n_H + (1 - y) \right) = -\frac{k_G n_H}{m + n_H} + (n_H - 1)(y + x) \frac{(k_B - 1)}{m} + (1 - y) + (y - x) \frac{1}{n_H}.\) Define the function \(M = \frac{(m + n_H)}{n_H} \left[ (n_H - 1)(y + x) \frac{(k_B - 1)}{m} + (1 - y) + (y - x) \frac{1}{n_H} \right].\) Now define \(n_{HM} = \arg \max(M),\) with \(0 < n_H < g(1 - \phi).\) If \(k_G > k_G3 = M(n_{HM}),\) then \(\frac{\partial G}{\partial \theta} + \frac{\partial H}{\partial \theta} < 0.\) This implies

\[
\frac{\partial n^*_H}{\partial c_p} < 0 \tag{39}
\]

On other hand, we have \(\frac{\partial n^*_i}{\partial c_p} = \frac{\partial f}{\partial c_p} = \begin{bmatrix}
\frac{1}{T} \\
\frac{1}{T}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial G}{\partial n_H} + \frac{\partial H}{\partial n_H} \\
\frac{\partial G}{\partial c_p} + \frac{\partial H}{\partial c_p}
\end{bmatrix}
\)

We know that \(\frac{\partial H}{\partial n_H} > 0\) (from (31)) and \(\frac{\partial G}{\partial n_H} < 0\) (from (33)). We have \(\frac{\partial G}{\partial n_H} + \frac{\partial H}{\partial n_H} = \theta(y - x) \left[ -\frac{1}{n_H} \right] + (1 - \theta) \frac{k_G m}{(m + n_H)} + \theta y \frac{(k_B - 1)}{m} - \frac{c_f}{T}.\) Define the function \(N = \frac{(m + n_H)^2}{(1 - \theta)m} \left[ \theta(y - x) \frac{1}{n_H} + \frac{c_f}{T} - \theta \frac{y(k_B - 1)}{m} \right].\) Define \((n_{H2}, \theta_{M2})\) that maximize \(N,\) with \(0 < n_H < g(1 - \phi)\) and \(0 < \theta < 1 - \phi.\) With
\[ k_G > k_{G1} = N(n_{H2}, \theta_{M2}), \text{ we have always } \frac{\partial G}{\partial n_H} + \frac{\partial H}{\partial n_H} > 0. \text{ So,} \]

\[ \frac{\partial \theta^*}{\partial c_p} < 0 \quad (40) \]

Define \( k_G > \tilde{k}_G = \max(k_{G3}, k_{G4}) \).

\[ \frac{\partial p^*_H}{\partial I} = \frac{\partial}{\partial I} \left[ \frac{1}{\tilde{p}_H} \right] = \frac{n_H - \frac{\partial n_H^*}{\partial I}}{(n_H)^2}. \text{ Let us determine the signal of } \frac{\partial n_H^*}{\partial I}. \text{ We have} \]

\[ \frac{\partial H}{\partial I} = \frac{c_p}{I^2} n_H \text{ (from (26 a)) and } \frac{\partial G}{\partial I} = \frac{c_f - c_p}{I^2} \text{ (from (26 b)).} \]

\[ \left[ \frac{\partial j}{\partial I} \right] = - \left[ \begin{array}{ccc} -\frac{\partial G \frac{c_p}{I^2}}{\partial n_H} & + \frac{\partial H \left[ \frac{c_f - c_p}{I^2} \right]}{\partial n_H} & \frac{c_f - c_p}{I^2} \\ -\frac{\partial G \frac{c_p}{I^2}}{\partial n_H} & - \frac{\partial H \left[ \frac{c_f - c_p}{I^2} \right]}{\partial n_H} & \frac{c_f - c_p}{I^2} \\ -\frac{\partial G \frac{c_p}{I^2}}{\partial n_H} & -\frac{\partial H \left[ \frac{c_f - c_p}{I^2} \right]}{\partial n_H} & \frac{c_f - c_p}{I^2} \end{array} \right] \]

So \( \frac{\partial j}{\partial I} = \frac{\partial n_H^*}{\partial I} = \frac{\frac{\partial G \frac{c_p}{I^2}}{\partial n_H} - \frac{\partial H \left[ \frac{c_f - c_p}{I^2} \right]}{\partial n_H}}{\frac{c_f - c_p}{I^2}} \).

We know that \( \frac{\partial G}{\partial \theta} > 0 \) (See (38)), \( \frac{\partial H}{\partial \theta} < 0 \) (see (30)) and \( \frac{\partial H}{\partial n_H} \frac{\partial G}{\partial \theta} - \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} > 0 \) (See (34)). This means that

\[ \frac{\partial n_H^*}{\partial I} < 0 \quad (41) \]

This implies that \( \frac{\partial p^*_H}{\partial I} > 0 \).

We also have \( \frac{\partial f}{\partial I} = \frac{\partial \theta^*}{\partial I} = \frac{\frac{\partial G \frac{c_p}{I^2} n_H - \frac{\partial H \left[ \frac{c_f - c_p}{I^2} \right]}{\partial n_H}}{\frac{c_f - c_p}{I^2}}}{\frac{c_f - c_p}{I^2}} \). We know that: \( \frac{\partial G}{\partial n_H} < 0 \) (see (33))

\[ \frac{\partial n_H^*}{\partial I} > 0 \text{ (see(31)). This implies that} \]

\[ \frac{\partial \theta^*}{\partial I} < 0 \quad (42) \]

\[ \frac{\partial p^*_H}{\partial k_G} = \frac{\partial}{\partial k_G} \left[ \frac{1}{\tilde{p}_H} \right] = -\frac{\partial n_H^*}{\partial I} \left( \frac{1}{n_H} \right)^2. \text{ Let us found the signal of } \frac{\partial n_H^*}{\partial k_G}. \text{ We have} \frac{\partial H}{\partial k_G} = \frac{(1-\theta)}{m+n_H} \]

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(from (26 a)) and \( \frac{\partial G}{\partial k_G} = 0 \) (from (26 b)).

\[
\left[ \begin{array}{c} \frac{\partial g}{\partial k_G} \\ \frac{\partial H}{\partial n_H} \end{array} \right] = - \left[ \begin{array}{cc} \frac{\partial G}{\partial \theta} & \frac{\partial G}{\partial n_H} \\ \frac{\partial H}{\partial \theta} & \frac{\partial H}{\partial n_H} \end{array} \right] \left[ \begin{array}{c} \frac{\partial G}{\partial \theta} \\ \frac{\partial H}{\partial \theta} \end{array} \right] = - \left[ \begin{array}{c} \frac{\partial G}{\partial \theta} \\ \frac{\partial H}{\partial \theta} \end{array} \right] \left[ \begin{array}{c} \frac{\partial G}{\partial \theta} \\ \frac{\partial H}{\partial \theta} \end{array} \right] \cdot (1 - \theta) \]

So, \( \frac{\partial g}{\partial k_G} = \frac{\partial n^*_H}{\partial k_G} = \frac{\partial G}{\partial n_H} \). We know that \( \frac{\partial G}{\partial \theta} > 0 \) (See (38)) and

\[
\frac{\partial H}{\partial n_H} \frac{\partial G}{\partial \theta} - \frac{\partial H}{\partial \theta} \frac{\partial G}{\partial n_H} > 0 \quad \text{(See (34))}. \]

This implies that

\[
\frac{\partial n^*_H}{\partial k_G} < 0 \quad \text{(43)}
\]

Consequently we have, \( \frac{\partial p^*_H}{\partial k_G} > 0 \).

We also have \( \frac{\partial f}{\partial k_G} = \frac{\partial \theta^*}{\partial k_G} = \frac{\partial G}{\partial n_H} \). Because \( \frac{\partial G}{\partial n_H} < 0 \) (See (33)), we have

\[
\frac{\partial \theta^*}{\partial k_G} < 0 \quad \text{(44)}
\]

### 4.5 Proof of Proposition 5

The equilibrium price \( p^*_H \) in the IPO subgame. We begin with (??):

\[
\theta (y - x) \left[ p_H - \frac{z}{m + n_H} V_B \right] = c_f - c_p, \quad \text{to obtain} \quad \frac{z}{m + n_H} V_B = p_H - \frac{c_f - c_p}{\theta (y - x)}. \]

Substituting this last equation in the equation (??) we have \( (1 - \theta) \left[ \frac{1}{m + n_H} V_G \right] + \theta y \left[ \frac{z}{m + n_H} V_B \right] - [1 - (1 - y) \theta] p_H - c_p = 0 \)
\[ p_H = \frac{1}{m + n_H} V_G - \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} - \frac{c_p}{(1 - \theta)} \]  \hspace{1cm} (45)

**The type Good entrepreneur.** The proportion of capital that the type Good entrepreneur gives up when go to public is \( f_G^* = \frac{n_H^*}{m + n_H^*} \). Note that the type good entrepreneur offers always the price \( p_H \), and so we know that the proportion of capital that he gives up to the investors is always the defined above.

The type Good entrepreneur is indifferent between the financing with VC and go public if \( s_g = f_G^* \), i.e., the proportion of capital that the entrepreneur gives up is equal in the two financing alternatives.

Beginning with (??) and substituing (45) we get:

\[
p_H^* n_H^* = I \iff \left[ \frac{1}{m + n_H^*} V_G - \frac{y (c_f - c_p)}{(1 - \theta^*)(y - x)} - \frac{c_p}{(1 - \theta^*)} \right] n_H^* = I
\]

\[
f_G^* = \frac{n_H^*}{m + n_H^*} = \frac{I}{V_G} + \frac{n_H^*}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta^*)(y - x)} + \frac{c_p}{(1 - \theta^*)} \right]
\]

(46)

Let us define \( F \) as follows:

\[
F = s_g^*(\rho, c_f) - f_G^* = s_g^*(\rho, c_f) - \frac{I}{V_G} - \frac{n_H^*}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta^*)(y - x)} + \frac{c_p}{(1 - \theta^*)} \right] = 0
\]

(47)

\( \rho_G(c_f, c_p) \) is defined implicitly by equation (47).

Next we are going to found the signal of \( \frac{\partial \rho_G}{\partial c_p} < 0 \) and \( \frac{\partial \rho_G}{\partial c_f} > 0 \).

\[
\frac{\partial \rho_G}{\partial c_p} = -\frac{\partial F}{\partial c_p} = -\left( \frac{\partial s_g^*}{\partial c_p} - \frac{\partial f_G^*}{\partial c_p} \right) = \frac{\partial f_G^*}{\partial \rho} \frac{\partial \rho}{\partial c_p}. \]

Notice that \( \frac{\partial s_g^*}{\partial c_p} = 0 \) and \( \frac{\partial s_g^*}{\partial \rho} > 0 \).
We have \( \frac{\partial f_G^*}{\partial c_p} = \frac{\partial}{\partial c_p} \left( \frac{f_G}{V_G} \right) \)
\[
= \frac{1}{V_G} \frac{\partial}{\partial c_p} \left[ \frac{y(c_f-c_p)}{(1-\theta^*)(y-x)} + \frac{c_p}{(1-\theta^*)} \right] + \frac{n_H^*}{V_G} \left[ \frac{(y-x)(y(c_f-c_p)\frac{\partial \theta^*}{\partial c_p} + (y-x)c_p\frac{\partial \theta^*}{\partial c_p}x(1-\theta^*))}{(1-\theta^*)(y-x)} \right]
\]

We know that \( \frac{\partial n_H^*}{\partial c_p} < 0 \) (see (39)) and that \( \frac{\partial \theta^*}{\partial c_p} < 0 \) (see (40)). This implies that

This implies that

\[
\frac{\partial F}{\partial c_p} = \frac{\partial f_G^*}{\partial c_p} < 0 \quad \text{and} \quad \frac{\partial \rho_G}{\partial c_p} < 0
\]

\[
\frac{\partial \rho_G}{\partial c_f} = -\frac{\partial F}{\partial c_f} = -\frac{\partial f_G^*}{\partial c_f} = \frac{\partial}{\partial c_f} \left( \frac{f_G}{V_G} \right)
= \frac{1}{V_G} \frac{\partial}{\partial c_f} \left[ \frac{y(c_f-c_p)}{(1-\theta^*)(y-x)} + \frac{c_p}{(1-\theta^*)} \right] + \frac{n_H^*}{V_G} \left[ \frac{y(y-x)(1-\theta^*)+y(y-x)(c_f-c_p)\frac{\partial \theta^*}{\partial c_f}+c_p\frac{\partial \theta^*}{\partial c_f}}{(1-\theta^*)(y-x)} \right]
\]

The signal of this expression is positive because \( \frac{\partial n_H^*}{\partial c_f} > 0 \) (see (35)) and \( \frac{\partial \theta^*}{\partial c_f} > 0 \) (see (??)). So, we have \( \frac{\partial s_f}{\partial c_f} > 0 \) and \( \frac{\partial f_G^*}{\partial c_f} > 0 \), which implies that the signal of \( \frac{\partial s_f}{\partial c_f} - \frac{\partial f_G^*}{\partial c_f} \) is not well defined.

We have \( \frac{\partial s_f}{\partial c_f} = \frac{1}{\sigma_f^2-2(\sigma_g^2+\sigma_f^2)\rho_{gf}} \) and \( \frac{\partial}{\partial \rho} \left( \frac{\partial s_f}{\partial c_f} \right) = \frac{2(\sigma_g^2+\sigma_f^2)(s_g^2+\rho_{gf})}{\sigma_f^2(\sigma_f^2-2(\sigma_g^2+\sigma_f^2)\rho_{gf})^2} > 0 \).

Consider \( \frac{\partial F(\rho_{V,c_f})}{\partial c_f} \), where \( \rho_V \) is the maximum value that \( \rho \) can assume (defined by the proposition 1). Now determine \( c_{fMax} = \text{argmax} \left( \frac{\partial F(\rho_{V,c_f})}{\partial c_f} \right) \), with \( 0 < c_f < c_f^B \). Where \( c_f^B \) is defined below. If \( \frac{\partial F(\rho_{V,c_f})}{\partial c_f} < 0 \), then \( \rho_M = \rho_V \).

If \( \frac{\partial F(\rho_{V,c_{fMax}})}{\partial c_f} > 0 \), then set \( \rho_1 \) (necessarily lower than \( \rho_V \), because \( -\frac{\partial}{\partial \rho} \left( \frac{\partial s_f}{\partial c_f} \right) > 0 \) such that \( \frac{\partial F(\rho_{c_{fMax}})}{\partial c_f} = 0 \), and define \( \rho_M \) equal to \( \rho_1 \). With \( \rho < \rho_M \), we have \( \frac{\partial F(\rho_{c_f})}{\partial c_f} < 0 \). This implies that
\[
\frac{\partial F}{\partial c_f} = \frac{\partial s_g^*}{\partial c_f} - \frac{\partial f_B^*}{\partial c_f} < 0 \quad \text{and} \quad \frac{\partial \rho_G}{\partial c_f} > 0
\] (49)

If the coefficient of risk-aversion is not too high, then an increase in the full cost has a great impact on the firm’s fraction to be given to the investors then on the fraction to be given to the VC.

**Type Bad entrepreneur.** If the type bad entrepreneur pools with the type good entrepreneur, he has to give up the firm fraction \( \frac{n_H}{m+n_L} \). On other hand if the bad entrepreneur separates from the good entrepreneur, he gives up the firm fraction \( \frac{n_H}{m+n_L} \). We know that, in equilibrium the bad entrepreneur is indifferent between pool and separate from the good entrepreneur, which means that, in equilibrium, this two fractions are equal.

Let us determine the fraction given in the separating hypotese. This fraction is comparable to the one given to the VC, because in the two situations the entrepreneur makes a full scale investment.

From equation (??) and (??), we have

\[
p_L n_L = I \iff \frac{1}{m+n_L} V_B n_L = I \iff \frac{n_L}{m+n_L} = \frac{I}{V_B} = f_B^*
\]

21 The type Bad entrepreneur is indifferent between the VC and the market if:

\[
T = s_f^*(\rho, c_f) - f_B^* = s_f^*(\rho, c_f) - \frac{I}{V_B} = 0
\] (50)

21 Note that the type good entrepreneur gives a capital proportion greater than \( \frac{I}{V_G} \). In fact, \( \frac{n_H}{m+n_H} = \frac{I}{V_G} + \frac{cn_H}{(1-\theta^eY^e)(1-Y)V_G} \). Why? Para se diferenciar?
Note that the fraction of capital that the entrepreneur has to give to the VC (defined in (25)) doesn’t depend on the cost of the information about the public period. This occurs because, with VC financing only the contract with full information belongs to the equilibrium.

The equation (50) defines implicitly $\rho_B(c_f)$. The last equation gives, for any $c_f$, the coefficient of risk-aversion that guarantees, for the type bad entrepreneur, the indifference between the VC financing and the public financing. We have $\frac{\partial \rho_B}{\partial c_f} = -\frac{\partial T}{\partial c_f} = -\frac{\partial s^*(\rho, c_f)}{\partial p}$. And so

$$\frac{\partial \rho_B}{\partial c_f} < 0$$ (51)

The risk-aversion of the VC is $\hat{\rho}$ (with $\hat{\rho} < \rho_V$, where $\rho_V$ comes from proposition 1). Let us define $c_f^B$ as the cost of full information that makes the type bad entrepreneur indifferent between the two financing alternatives. $c_f^B$ must solve $\rho_B(c_f)=\hat{\rho} \Leftrightarrow s^*_f(\hat{\rho}, c_f) - \frac{I}{V_B} = 0$. Notice that this solution doesn’t depend on $c_p$.

Given the VC risk-aversion, the type good entrepreneur is indifferent between going public and the VC if:

$$\rho_G(c_f, c_p) = \hat{\rho} \Leftrightarrow F = s^*_g(\hat{\rho}, c_f) - \frac{I}{V_G} - \frac{n_G}{V_G} \left[ \frac{y(c_f-c_p)}{(1-\theta^*)(y-x)} + \frac{c_p}{(1-\theta^*)} \right] = 0$$

$$\Leftrightarrow s^*_g(\hat{\rho}, c_f) - \frac{I}{V_G} - \frac{n_G}{V_G} \left[ \frac{yc_f}{(1-\theta^*)(y-x)} - \frac{xc_p}{(1-\theta^*)(y-x)} \right] = 0.$$  

Solving this equation in $c_p$ we get:
\[
c_p = \frac{(1-\theta^*)(y-x)V_G}{xn_H^*} \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) \right] + \frac{y}{x}c_f \quad (52)
\]

Given \(\bar{\rho}\), we have a set of \((c_p, c_f)\), that guarantees the indifference between the VC and going public. In respect to this function, is possible to demonstrate that: (i) if \(c_f = 0\), then \(c_p < 0\) (point X- see Figure (?? a)) (ii) if \(c_p = 0\), then \(c_f > 0\) (point Y).

First, if \(c_f = 0\), then

\[
c_p = \frac{(1-\theta^*)(y-x)V_G}{xn_H^*} \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, 0) \right].
\]

From (?? a), we have

\[
s_{gf}^*(\bar{\rho}, c_f) = \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) = I - c_f \leftrightarrow s_{gf}^* = \frac{I}{V_G} + c_f + \frac{\bar{\rho} s_{gf}^2(\sigma_{gf}^2 + \sigma_{\epsilon}^2)}{V_{gf}}.
\]

This means that \(s_{gf}^* > \frac{I}{V_G}\). Because \(V_G > V_{gf}\), we have \(s_{gf}^* > \frac{I}{V_G} > \frac{I}{V_{gf}}\). So, we have

\[
c_p = \frac{(1-\theta^*)(y-x)V_G}{xn_H^*} \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) \right] < 0.
\]

This implies that \(c_f = -(1-\theta^*)(y-x)V_G \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) \right] > 0\)

Second, if \(c_p = 0\), then we have:

\[
0 = \frac{(1-\theta^*)(y-x)V_G}{xn_H^*} \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) \right] + \frac{y}{x}c_f \iff c_f = \frac{(1-\theta^*)(y-x)V_G}{yG^*} \left[ \frac{I}{V_G} - s_{gf}^*(\bar{\rho}, c_f) \right] > 0.
\]

Assuming that we are in a point A belonging to the curve \(\rho_G(c_f, c_p) = \bar{\rho}\), where \(F = 0\) (figure ?? b). If \(c_f\) increase, then F decrease, because \(\frac{\partial F}{\partial c_f} < 0\) (see(49)). We are now in point B. In order to return to the curve, F must return to 0 (\(\Delta F > 0\)), which implies that \(c_p\) must increase because \(\frac{\partial F}{\partial c_p} > 0\) ((48)). We are now in point C. This implies that \(\rho_G(c_f, c_p) = \bar{\rho}\) has always a positive inclination - for simplicity we draw a linear function.

Let us see the individual choice of different entrepreneurs, in function of \(c_p\) and \(c_f\). With \(c_f^B\) the type bad entrepreneur is indifferent between
Figure 5:

the VC and going public. Consider now a $c_f$ lower than $c_f^B$, but maintaining $\rho$. \[ \frac{\partial T}{\partial c_f} = \frac{\partial s_f^*}{\partial c_f} = \frac{x}{1-x} \frac{\partial s_h^*(c_f, \rho)}{\partial c_f} > 0, \] from proposition 2. With $c_f < c_f^B$, we have $T < 0 \iff s_f^*(\rho, c) < f_B^*$, which means that the type bad entrepreneur choose the VC.

In relation to the type good entrepreneur, consider $\tilde{c}_f$ and $\tilde{c}_p$ such that $F = 0$ (they represent any point in the curve $\rho_G(c_p, c_f) = \tilde{\rho}$ (for example point A in figure (?? a)). We have $\frac{\partial F}{\partial c_p} = \frac{\partial s_f^*}{\partial c_p} - \frac{\partial f_G^*}{\partial c_p} = -\frac{\partial f_G^*}{\partial c_p} > 0$ (see(48)). With $\tilde{c}_f$ and $c_p > \tilde{c}_p$ (Point B), then we have $F > 0$. This means that the type good entrepreneur choose going public. With $\tilde{c}_f$ and $c_p < \tilde{c}_p$ (Point C), we have $F < 0$ meaning that the type good entrepreneur choose VC.

On other hand, we have $\frac{\partial F}{\partial c_f} = \frac{\partial s_f^*}{\partial c_f} - \frac{\partial f_G^*}{\partial c_f} < 0$ (see 49). When we have $\tilde{c}_p$ and $c_f > \tilde{c}_f$ (Point D in figure (?? a)), this means $F < 0$. So, the type good entrepreneur choose the VC. On reverse, with $\tilde{c}_p$ and $c_f < \tilde{c}_f$ (Point E) the type good entrepreneur choose going public.
Consider the case $c_f < c_f^B$. To the right of the curve $\rho_G(c_f, c_p) = \tilde{\rho}$, both the type good and type bad entrepreneur choose the VC (see Figure (?? a)). To the left of the curve $\rho_G(c_f, c_p) = \tilde{\rho}$ the type good entrepreneur choose going public and the type bad entrepreneur choose (without considering the type good entrepreneur choice) the VC. In this situation and given the out-of-equilibrium beliefs, if the type bad entrepreneur choose the VC, he will be identified as type bad. So, the VC will demand a share $s_B^*$ (see (??)), corresponding to a situation were the VC identifies the type bad firm, even without information: $s_B^*V_B - \rho s_B^*2(0 + \sigma^2_\varepsilon) = I + 0 \iff s_B^* = \frac{I}{V_B} + \frac{\rho s_B^*2\sigma^2_\varepsilon}{V_B} > \frac{I}{V_B}$.

Hence, the type bad entrepreneur prefers to go public, because $f_B^* = \frac{I}{V_B} < s_B^*$.

Consider $\tilde{\rho}$ and $\tilde{c} = c_f = c_p$. With these costs and this coefficient of risk aversion, the two entrepreneurs are, simultaneously, indifferent between the VC and going public. First, we obtain $\tilde{c}$ solving $\rho_B(\tilde{c}) = \rho_G(\tilde{c}, \tilde{c})$ (represented
in point A in figure (?? b)). In second, we obtain \( \hat{\rho} \) solving \( \rho_B(\hat{c}) = \hat{\rho} \).

For the type bad entrepreneur initially we have \( \rho_B(\hat{c}) = \hat{\rho} \). Now consider a \( \rho < \rho_N = \min(\hat{\rho}, \rho_M) \) where \( \rho_M \) is defined above. With \( \rho_G \) we have \( \rho_B(\hat{c}) > \rho_{VC} \). Because \( \frac{\partial \rho_B}{\partial c_f} < 0 \) (see (51)), \( c_f \) must increase in order to \( \rho_B \) decrease. This implie that \( c_f^B \) moves to the right.

For the type good entrepreneur initially we have \( \rho_G(\hat{c}, \hat{c}) = \hat{\rho} \). With \( \rho \) we have \( \rho_G(\hat{c}, \hat{c}) > \rho \). Because \( \frac{\partial \rho_G}{\partial c_f} > 0 \) and \( \frac{\partial \rho_G}{\partial c_p} < 0 \), \( c_f \) must decrease and/or \( c_p \) must increase in order to \( \rho_G \) decrease. In conclusion, the curve \( \rho_G(c_f, c_p) = \rho \) is to the left of \( \rho_G(c_f, c_p) = \hat{\rho} \) (see figure (?? b)).

Next, we will prove that there is no equilibrium with \( c_f > c_f^B \). Above \( c_f^B \), the type good entrepreneur choose the VC and the type bad entrepreneur choose the public financing. So, the type bad entrepreneur is identified as such by the investors. This implies that no investor will buy shares at price \( p_H \) to this firm. Hence, the firm is better of if offers always the low price \( (\beta = 0) \), because it can sell all the offered shares. In conclusion, the type bad will choose the public financing, offering always the price \( p_L \), and the investors do not produce information. The type good firm choose the VC. Because the firms reveal its own type, the VC does not need to produce information.

Without information production by the VC, the type bad firm gains in desviating to the VC? In the above situation, the type good firm is offering
to the VC the fraction $s^*_G$, that solves: $s_G V_G - \rho s^*_G (\sigma^2_{eD} + \sigma^2_{eI} + 2 \sigma_{DI}) = I$.

The type bad firm is offering to the investors the fraction of capital $I V_B$.

We know that, when $c_f = 0$ the type bad firm choose the VC financing: $s^*_f(0, \rho) < \frac{I}{V_B}$. We have $s^*_f(0, \rho) = x s^*_g(0, \rho) + (1-x) s^*_b(0, \rho)$. $s^*_g(0, \rho)$ solves $s^*_g V_g - \rho s^*_g (\sigma^2_g + \sigma^2_{eD} + 2 \sigma_{DI}) = I$ and $s^*_b(0, \rho)$ solves $s^*_b V_b - \rho s^*_b (\sigma^2_{eD} + \sigma^2_{eI} + 2 \sigma_{DI}) = I$. Because $V_g < V_G$ and $(\sigma^2_g + \sigma^2_{eD} + 2 \sigma_{DI}) < (\sigma^2_g + \sigma^2_{eD} + \sigma^2_{eI} + 2 \sigma_{DI})$, we have $s^*_G < s^*_g(0, \rho)$, and because $V_b < V_G$ we have $s^*_G < s^*_b(0, \rho)$. This means that $s^*_G < s^*_f(0, \rho)$. This implies that $s^*_G < \frac{I}{V_B}$, meaning that the type bad firm has advantage in desviating to the VC financing. In reaction, the VC begin to produce information, and the type bad firm choose again the public financing. So, there is no equilibrium with $c_f > c_f^B$.

Now we have to consider the limitations imposed by the proposition 1 and 3. We must have $c_f^U = \min(c_f^B, c_{fV})$, where $c_{fV}$ comes from the proposition 1. From proposition 3, we have two additional restrictions: (1) $c_f \frac{(1-y)}{(1-x)} \geq c_p$ and (2) $c_p + d_s > c_f$. Notice that this two conditions only limit the going public equilibrium. The condition (1) limits always the going public equilibrium. The condition (2) only limits the going public equilibrium if $d_S < c_f^G$, where $c_f^G$ solves $\rho_G(c_f, 0) = \rho$. One possible equilibrium, where condition (2) is not active, is described by the Figure (??).
Figure 7:

4.6 Proof of Corollary 6

Let us begin with the equation (47): $F = s_{gf}^*(\rho, c_f) - f_G^* = s_{gf}^*(\rho, c_f) - \frac{f}{V_G} - n \cdot H \cdot y (c_f - c_p) (1 - \theta) (y - x) + c_p (1 - \theta) \cdot \theta$$\partial \theta^* \partial I \cdot (1 - \theta)^2 + c_p \cdot \theta^* \partial I \cdot (1 - \theta)^2$.

$\frac{n}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} + \frac{c_p}{(1 - \theta)} \right] = 0$. $\rho_G$ is implicitly defined by this equation.

Proof for $I$

$\partial \rho_G \partial I = -\frac{\partial F}{\partial I} \frac{\partial s_{gf}^*}{\partial I} - \frac{\partial f^*_G}{\partial I}$. We have $\frac{\partial s_{gf}^*}{\partial I} < 0$ and $\frac{\partial f^*_G}{\partial I} > 0$. $\frac{\partial \rho^*_G}{\partial I} =$

$\frac{\partial}{\partial I} \left[ \frac{n}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} + \frac{c_p}{(1 - \theta)} \right] \right] = \frac{\partial}{\partial I} \left[ \frac{n}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} + \frac{c_p}{(1 - \theta)} \right] \right]$

$= -k_G \left[ \frac{n}{V_G} \left[ \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} + \frac{c_p}{(1 - \theta)} \right] \right] + \frac{1}{V_G} \left[ \frac{\partial n}{\partial I} \left[ \frac{y (c_f - c_p)}{(1 - \theta)(y - x)} + \frac{c_p}{(1 - \theta)} \right] \right] + n_H^* \left[ \frac{y (c_f - c_p) (y - x) \theta^*}{(1 - \theta^*)^2} + c_p \cdot \theta^* \cdot \theta^* \right]$

We know that $\frac{\partial n^*_H}{\partial I} < 0$ (see (41)) and $\frac{\partial \rho^*}{\partial I} < 0$ (see (42)). This means that $\frac{\partial f^*_G}{\partial I} < 0$. This implies that $\left[ \frac{\partial s_{gf}^*}{\partial I} - \frac{\partial f^*_G}{\partial I} \right]$ has an undifined signal.

Consider $\frac{\partial s_{gf}^*}{\partial I} = -\frac{\partial F}{\partial I} \frac{\partial s_{gf}^*}{\partial I} = \frac{-s_{gf}^* \frac{\partial V_G}{\partial I} + 1}{V_G - 2(\sigma_{gf}^2 + \sigma_{gf}^2) \rho s_{gf}^*}$ and
\[
\frac{\partial}{\partial \rho} \left[ \frac{s_{gf}^*}{\partial I} \right] = \left( -\frac{\partial V_{gf}}{\partial I} \right) \left( s_{gf}^* \right)^2 \left( V_{gf} - 2(\sigma_{gf}^2 + \sigma_e^2) \rho s_{gf}^* \right) \left( -s_{gf}^* \frac{\partial V_{gf}}{\partial I} + 1 \right) \left( -2(\sigma_{gf}^2 + \sigma_e^2) \rho s_{gf}^* \right) \left( s_{gf}^* + \rho \frac{\partial s_{gf}^*}{\partial \rho} \right) \left( V_{gf} - 2(\sigma_{gf}^2 + \sigma_e^2) \rho s_{gf}^* \right)^2
\]

From (??) we have \( -s_{gf}^* \frac{\partial V_{gf}}{\partial I} + 1 \) < 0. This means that \( \frac{\partial}{\partial \rho} \left[ \frac{s_{gf}^*}{\partial I} \right] \) < 0. \( \frac{\partial s_{gf}^*}{\partial I} \) is the lower possible when \( \rho \) is the maximum possible: \( \rho = \rho_V \) (defined in proposition 1).

Now determine \( c_{fMin} = \text{argmin} \left( \frac{\partial F(\rho_V, c_f)}{\partial I} \right) \), with \( 0 < c_f < c_f^B \). Where \( c_f^B \) is defined in proposition 5. If \( \frac{\partial F(\rho_V, c_f)}{\partial I} > 0 \), then \( \rho_{M2} = \rho_V \). If \( \frac{\partial F(\rho_V, c_f)}{\partial I} < 0 \), then found \( \rho_2 \) (necessarily lower then \( \rho_V \), because \( \frac{\partial s_{gf}^*}{\partial \rho} < 0 \)) such that \( \frac{\partial F(\rho, c_f)}{\partial I} = 0 \), and define \( \rho_{M2} = \rho_2 \). With \( \rho < \rho_{M2} \), we have \( \frac{\partial F(\rho, c_f)}{\partial I} > 0 \), which implies that \( \frac{\partial \rho_{M2}}{\partial \rho} < 0 \).

Initially we are in a point A were \( \rho_G(c_p, c_f) = \rho \) (see figure (8)). With the increase in I, we have \( \rho_G(c_p, c_f) < \rho \). In order to restore the equality, \( \rho_G(c_p, c_f) \) must increase, throughout the decrease of \( c_p \) \( \left( \frac{\partial \rho_G}{\partial c_p} < 0 \right) \) and/or a increase in \( c_f \) \( \left( \frac{\partial \rho_G}{\partial c_f} > 0 \right) \). For example, we are now in point B. This means that the curve \( \rho_G(c_p, c_f) = \rho \) has moved to the right.

**Proof for** \( \sigma^2_e : \frac{\partial \rho_G}{\partial \sigma^2_e} = -\frac{\partial F(\rho_G, c_f)}{\partial c_f} = -\left( \frac{\partial s_{gf}^*}{\partial \rho} \frac{\partial \sigma^2_e}{\partial \sigma^2_e} \right) = -\frac{\partial \sigma^2_e}{\partial \rho} < 0 \). Notice that \( \sigma^2_e \) doesn’t affect \( n_{gf}^* \) and \( \theta^* \) (see equation (26)). This imply, like for I, that an increase in \( k_G \) moves the curve \( \rho_G(c_p, c_f) = \rho \) to the right.
4.7 Proof of Corollary 7

\[
\frac{\partial \rho}{\partial k_G} = -\frac{\partial F}{\partial k_G} = -\frac{\partial s_f}{\partial k_G} = -\frac{\partial F}{\partial s_f} = -s_f \frac{\partial V_f}{\partial k_G} < 0
\]

and \( \frac{\partial s_f}{\partial \rho} > 0 \).

\[
\frac{\partial f}{\partial k_G} = \frac{\partial}{\partial k_G} \left( 1 + \frac{n_H}{k_G} \left[ y(c_f - c_p) + \frac{c_p}{1-th^*} \right] \right) = -\frac{1}{k_G} \left( 1 + \frac{n_H}{k_G} \left[ y(c_f - c_p) + \frac{c_p}{1-th^*} \right] \right)

+ \frac{1}{k_G} \left( \frac{\partial n_H}{\partial k_G} \left( y(c_f - c_p) + \frac{c_p}{1-th^*} \right) \right)

We know that \( \frac{\partial n_H}{\partial k_G} < 0 \) (see (43)) and \( \frac{\partial s_f}{\partial k_G} < 0 \) (see (44)). This means that \( \frac{\partial f}{\partial k_G} < 0 \). So, the signal of \( \frac{\partial \rho}{\partial k_G} \) is undefined.

We have

\[
\frac{\partial}{\partial \rho} \frac{\partial s_f}{\partial k_G} = \left( \frac{-\partial V_f}{\partial k_G} \frac{\partial s_f}{\partial \rho} \right) \left( V_f - 2(\sigma_f^2 + \sigma_f^2) \rho_{s_f}^2 \right) - \left( -s_f \frac{\partial V_f}{\partial k_G} \right) \left[ -2(\sigma_f^2 + \sigma_f^2) \left( s_f^2 + \rho \frac{\partial s_f}{\partial \rho} \right) \right] < 0
\]

So, \( \frac{\partial s_f}{\partial k_G} \) is the lower possible when \( \rho \) is the maximum possible: \( \rho = \rho_V \) (defined in proposition 1). Now determine \( c_{fMin}' = \arg\min \left( \frac{\partial F(\rho_V, c_f)}{\partial k_G} \right) \), with
0 < c_f < c_f^U. Where c_f^U is defined in proposition 5. If \( \frac{\partial F(\rho, c_f')}{\partial k_G} \) > 0, then \( \rho_{M3} = \rho_V \). If \( \frac{\partial F(\rho, c_f')}{\partial k_G} \) < 0, then found \( \rho_3 \) (necessarily lower then \( \rho_V \), because \( \frac{\partial F(\rho, c_f')}{\partial k_G} \) < 0) such that \( \frac{\partial F(\rho, c_f')}{\partial k_G} = 0 \), and define \( \rho_{M3} = \rho_3 \). With \( \rho < \rho_{M3} \), we have \( \frac{\partial F(\rho, c_f)}{\partial k_G} > 0 \), which implies that \( \frac{\partial \rho_G}{\partial k_G} < 0 \). This means, like in proposition 6, that an increase in \( k_G \) moves the curve \( \rho_G(c_p, c_f) = \rho \) to the right.