On the Hedging Strategy of a European Option with Discrete Stochastic Dividends

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Abstract

In this note we value and hedge European options on assets paying discrete stochastic dividends. Our hedging strategy is based only on the underlying asset, risk-free bonds and dividend strips. Our simple strategy is easily seen to be compatible with early results based, among other things, on the existence of a dividend forward contract. Such contracts, however, are not traded in the market place and, therefore, our simpler strategy gains in clarity and practical application.

In this note we analyze the problem of valuing a European call option written on an asset paying discrete random dividends. Dividends are an important feature when characterizing the process of the underlying asset for the purpose of pricing derivatives, as stressed in the early papers of Merton (1973) and Black (1975). In Black (1975) the dividends were considered to be stochastic, although with a continuous deterministic yield, whereas in Merton (1973) dividends are discrete, paying a known amount at a given payment date. A paper by Chance, Kumar and Rich (2002) [Chance *et al.* hereafter] relaxes these assumptions, allowing for stochastic, discrete dividends.

As usual, the valuation of a European option is given by the value of a self-financing portfolio that replicates the final payoff of the option at maturity. The application of this principle to random dividends is an important

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development for the practitioners in the derivatives market, for two reasons. First, for realism. Dividends are very often not certain and contingent in the state of Nature. Second, for risk management purposes, since the valuation of the option implies to make explicit the dynamic strategy that fully hedges the risky positions on the option.

In order to construct such hedging portfolios, Chance *et al.* (2002) postulate, among other things, the existence of a dividend forward contract, namely a forward contract in which the underlying asset is the future dividend to be paid by the asset on which the option is written. However, these authors recognize that they are not aware of transactions of such instruments in today's markets. Also, following and extending Brennan (1998), these authors assume that dividends on individual stocks on the S&P 500 can be stripped and sold as separate claims. Given the random nature of the dividends, the hedging portfolio proposed involves not only the usual underlying stock and riskless bonds, but also these dividend forward contracts.

Because of the practical relevance of the hedging portfolio, and since the dividend forward contracts are not traded, in this note we propose an equivalent but simpler hedging strategy that does not include such forward contracts. In fact, as Chance et al. (2002) recognize in their conclusion, such contracts are purely redundant in the presence of dividend strips. We propose to completely erase the concept of dividend forward contracts and keep only the concept of dividend strip, incorporating it in the hedging strategy.

In order to offer a result-oriented analysis and provide a full treatment of the mathematical and statistical information on the techniques, for the purpose of illustration we first solve the traditional case early suggested by Roll (1977) in the first section. The methodology is thus extended to the case of random dividends in the second section. A concluding section comments some implications.

1 Non-Stochastic Dividends

Consider an economy with a risk-free interest rate r and a risky asset with price process S_t and volatility σ. Let S_0 denote the value of the underlying asset at $t = 0$. For a European call option written on that asset, with exercise price K and maturity T, consider the case where the asset pays a discrete dividend D at time $\tau < T$.

1.1 Assumption

In the non-stochastic case the following assumption is made: Take $I_{[0,\tau[}(t) = 1$ if $t \in [0,\tau[$ and zero otherwise. Then, for all $t \in [0,T]$, the process $\tilde{S}_t = S_t - De^{-r(\tau-t)} I_{[0,\tau[}(t)$ is a Geometric Brownian motion with initial value $\tilde{S}_0 = S_0 - De^{-r\tau}$. In other words, the process $\tilde{S}_t$ follows the stochastic differential equation

$$\begin{aligned} d\tilde{S}_t &= r\tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ \tilde{S}_t &= S_t - De^{-r(\tau-t)}, \end{aligned}$$

where W_t is a Wiener process with respect to the risk-neutral probability measure. As argued by Roll (1977), under this assumption, the value of the call is given by the Black-Scholes formula, replacing S_0 by $\tilde{S}_0$. This means that the value of the call is

$$C_t = \left(S_t - De^{-r(\tau-t)}\right) N\left[d_1\right] - Ke^{-r(T-t)} N\left(d_2\right) \tag{1}$$

where

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du \\ d_1 &= \frac{\ln\left(\frac{S_t - De^{-r(\tau-t)}}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

1.2 The Hedging Portfolio

In this section we build a portfolio in order to hedge the risk of a long position on the call option. One such portfolio should be self-financing, *i.e.*, does not require further financing beyond the initial investment [see Nielsen (1999)], and should replicate the value of the call at maturity. It then follows that it replicates the option at any point in time and thus, the value of the call option at arbitrary t should equal the cost of such portfolio at that point in time.

In order to construct one such portfolio, we assume that the time-zero value of a zero-coupon bond is normalized to one. At time t, the value of a zero-coupon bond is thus given by

$$B_t = \exp(rt). \tag{2}$$

Consider the following strategy. For any t buy $\phi_1 = N(d_1)$ shares of stock and buy $\phi_2 = -Ke^{-rT}N(d_2) - De^{-r\tau}I_{[0,\tau[}(t)\,N(d_1)$ zero-coupon bonds with

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{\tilde{S}_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2 &= d_1 - \sigma\sqrt{T-t}
\end{aligned}
$$

Let V_t denote the value of such portfolio. Then

$$
V_t = S_t N(d_1) - \left[Ke^{-rT}N(d_2) + De^{-r\tau}I_{[0,\tau[}(t)\,N(d_1)\right]B_t \tag{3}
$$

Proposition 1 *The value of the option at any point in time is given by the corresponding cost of such strategy.*

Proof. We need to show that the above hedging portfolio is continuous at τ, self-financed and replicates the value of the call option at maturity. Then the result follows.

Regarding continuity, let $\tau^- = \lim_{t\nearrow\tau}$. Then $S_\tau = S_{\tau^-} - D$. Use of (2) in equation (3) leads to

$$
V_{\tau^-} = (S_{\tau^-} - D)\,N(d_1) - Ke^{-rT}N(d_2)\,B_\tau = V_\tau.
$$

In order to be self-financed, we must show that the portfolio satisfies $dV_t = \phi_1 dS_t + \phi_2 dB_t$. Using Ito's lemma

$$
dV_t = N(d_1)\,d\tilde{S}_t - Ke^{-rT}N(d_2)\,dB_t
$$

Noticing that $\tilde{S}_t = S_t - De^{-r(\tau-t)}I_{[0,\tau[}(t)$, then $d\tilde{S}_t = dS_t - De^{-r\tau}I_{[0,\tau[}(t)\,dB_t$ and

$$
\begin{aligned}
dV_t &= N(d_1)\,d\tilde{S}_t - Ke^{-rT}N(d_2)\,dB_t \\
&= N(d_1)\left[dS_t - De^{-r\tau}I_{[0,\tau[}(t)\,dB_t\right] - Ke^{-rT}N(d_2)\,dB_t \\
&= N(d_1)\,dS_t - \left[N(d_1)\,De^{-r\tau}I_{[0,\tau[}(t) + Ke^{-rT}N(d_2)\right]dB_t \\
&= \phi_1 dS_t + \phi_2 dB_t.
\end{aligned}
$$

Finally, the replication of the value of the call for $t \in [\tau, T]$ is straightforward, since in that region the hedging strategy corresponds to the traditional Black-Scholes specification. ■

2 Stochastic Dividends

In what follows, we reproduce the analysis above for the case where the dividends to be paid at time τ are random. We follow the framework of Chance *et al. (2000)* and their assumptions.

2.1 Assumptions

In the case of stochastic dividends, Chance *et al. (2002)* make the following assumptions

1. Following Brennan (1998), dividends on the S&P 500 can be stripped and sold as separate claims. The present value of expected future dividend at any time $t < \tau$ is observable and specified by

   $$D_t(\tau) = E[D_\tau|\mathcal{F}_t]\, e^{-k(\tau - t)},$$

   the value at t of the dividend strip. If one such claim cannot be directly transacted, we assume that, at least, it can be synthetically reproduced in the market.

2. We assume that $\lim_{t \nearrow \tau} E[D_\tau|\mathcal{F}_t] = D_\tau$.[1]

3. The process $\tilde{S}_t = S_t - D_t(\tau)\, I_{[0,\tau[}(t)$ is a Geometric Brownian motion with initial value $\tilde{S}_t = S_t - D_t(\tau)$. In other words,

   $$\begin{aligned} d\tilde{S}_t &= r\tilde{S}_t dt + \sigma \tilde{S}_t dW \\ \tilde{S}_t &= S_t - D_t(\tau). \end{aligned}$$

4. There exists a forward contract on the underlying stock.

5. There exists a dividend forward contract, namely a forward contract in which the underlying asset is the dividend at time τ.

Under these assumptions Chance *et al.* (2002) show that the Black-Scholes formula remains valid with $\tilde{S}_t = S_t - D_t(\tau)$, *i.e.*,

$$C_t = N(d_1)[S_t - D_t(\tau)] - Ke^{-rT} N(d_2)\, B_t \tag{4}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S_t - D_t(\tau)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}} \\ d_2 &= d_1 - \sigma\sqrt{T - t}. \end{aligned}$$

[1] Although Chance *et al* (2002) do not explicitly assume this limit, this is a necessary condition for their endnote 11 to hold.

2.2 The Hedging Portfolio

In this section we provide a hedging strategy that replicates C_t in (4) at maturity, is self-financing and continuous in time.

Consider the following strategy. At any time t, buy $\phi_1 = N(d_1)$ shares and $\phi_2 = -Ke^{-rT}N(d_2)$ zero-coupon bonds. Furthermore, if $t < \tau$, short ϕ_1 dividend strips. Notice that

$$d_1 = \frac{\ln\left(\frac{\tilde{S}_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t},$$

but the process $\tilde{S}_t$ is defined in a subtly different way from the former section. Let V_t denote the value of such portfolio. Then

$$V_t = \phi_1 S_t - \phi_1 D_t(\tau) I_{[0,\tau[}(t) + \phi_2 B_t$$

Proposition 2 *The value of the option at any point in time is given by the corresponding cost of such strategy.*

Proof. Again, we need to show that the above hedging portfolio is continuous at τ, self-financed and replicates the value of the call option at maturity.

Regarding continuity,

$$\begin{aligned} V_{\tau^-} &= N(d_1)S_{\tau^-} - N(d_1)D_{\tau^-}(\tau) - Ke^{-rT}N(d_2)B_{\tau^-} \\ &= N(d_1)\left(S_{\tau^-} - E\left[D_\tau|\mathcal{F}_{\tau^-}\right]e^{-k\left(\tau-\tau^-\right)}\right) - Ke^{-rT}N(d_2)B_{\tau^-} \\ &= N(d_1)(S_{\tau^-} - D_\tau) - Ke^{-rT}N(d_2)B_{\tau^-} = V_\tau \end{aligned}$$

where the third line is obtained with the use of Assumption 2 and the final result comes from continuity of B_t and the fact that $S_\tau = S_{\tau^-} - D_\tau$.

Regarding the self-financing property, under Assumption 3 we use Ito's lemma as in the procedure with non-stochastic dividends to obtain

$$dV_t = N(d_1)\,d\tilde{S}_t - Ke^{-rT}N(d_2)\,dB_t.$$

Noticing that $d\tilde{S}_t = dS_t - dD_t(\tau) I_{[0,\tau[}(t)$,

$$\begin{aligned} dV_t &= N(d_1)\,d\tilde{S}_t - Ke^{-rT}N(d_2)\,dB_t \\ &= N(d_1)\left[dS_t - dD_t(\tau) I_{[0,\tau[}(t)\right] - Ke^{-rT}N(d_2)\,dB_t \\ &= N(d_1)\,dS_t - \left[N(d_1)De^{-r\tau}I_{[0,\tau[}(t) + Ke^{-rT}N(d_2)\right]dB_t \\ &= \phi_1 dS_t + \phi_1 dD_t(\tau) I_{[0,\tau[}(t) + \phi_2 dB_t. \end{aligned}$$

thus proving that the strategy is self-financing. The replication of the value of the call at maturity is also straightforward, since for $t > \tau$ the strategy corresponds to the traditional Black-Scholes specification. The proposition follows. ■

3 Concluding Remarks

As it can be easily seen, we used only three of the five assumptions underlying the results in Chance *et al.* (2002). We used Assumption 1 when shorting dividend strips building the hedging strategy. Assumption 2 was used when discussing continuity. Finally, Assumption 3 was used to justify the Black-Scholes formula for the price $\tilde{S}_t$ and to prove that the hedging strategy was self-financed.

However we did not make any use of either Assumptions 4 and 5, for they are redundant given the first three Assumptions. In fact, in Chance *et al. (2002)*, the use of dividend forward contracts may look necessary to replicate the dividend strip. In that case, dividend strips would be redundant given dividend forward contracts. However, in order to define the arbitrage-free value of these forwards, a long position in dividend strip is required in Chance *et al. (2002)*. Hence, it makes sense to use dividend forward contracts in an arbitrage-free context, only in the presence of dividend strips. Thus, our hedging strategy is clearly both simpler and more realistic.

References

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