Exact Limit of the Expected Periodogram in the Unit-Root Case

João Valle e Azevedo*

Banco de Portugal & Universidade NOVA de Lisboa

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Abstract

We derive the limit of the expected periodogram in the unit-root case under general conditions. This function is seen to be time-invariant at frequencies other than zero (i.e., it does not depend on initial conditions and it does not change with time), thus sharing a fundamental property with the stationary case equivalent. However, defining this function as the (pseudo-) spectrum invalidates the usual interpretation of the consequences of applying linear filters that render the original (integrated of order 1) series stationary. That is, the spectrum of the filtered series is no longer the transfer function of the filter multiplied by the (pseudo-) spectrum of the original series. This conclusion does not hold if the order of integration is $1 + d$, with $-0.5 < d < 0$ (a non-stationary environment).

Keywords: Periodogram, Unit root, Pseudo-Spectrum

*Address: Av. Almirante Reis, 71-6th floor, 1150-012 Lisboa, Portugal; E-mail: azevedojv@gmail.com; Phone: +351 213130163. Most research was done while I was a graduate student at Stanford University. I gratefully acknowledge the Portuguese Government, through the Fundação para a Ciência e Tecnologia, for financial support during my graduate studies.
1 Motivation

Solo (1992) has shown that certain continuous-time stationary increment processes possess many of the frequency domain properties of stationary processes. Crucially, although their variance is infinite or time-varying (depending on the specification of initial conditions), they have a time-invariant spectrum, defined there as the limit of the expected periodogram. This more general definition of spectrum helps us understand the frequency domain properties of certain non-stationary processes, circumventing the restrictive nature of the standard spectral representation theorems for stationary processes. Crucially, and this is our main concern, it sheds light on the frequency domain interpretation of the effects of applying trend extraction filters to integrated time series. Some of these filters (e.g., the Baxter and King (1999) filter or the Hodrick and Prescott (1997) filter) are routinely employed to (supposedly) isolate specific frequencies (e.g., the so-called business cycle frequencies) in integrated macroeconomic time series. Also, comparing moments of filtered (observed) integrated data with moments of (filtered) data generated by real business cycle models or by new-Keynesian general equilibrium models is a standard exercise since Kydland and Prescott (1982). Now, in order to interpret the isolated fluctuations (or the transfer function of the filters) we need to characterize the frequency domain properties of integrated time series.

We show in this note that Solo’s (1992) main result holds in the case of (discrete-time) time series processes containing one unit root. Under very general conditions, we provide exact expressions for the time-invariant spectrum of an integrated time series, defined as the limit of the expected periodogram. It is shown that this limit differs from the commonly defined (pseudo-) spectrum of an integrated (of order 1) time series (e.g., as in Harvey (1993), Hurvich and Ray (1995), Young, Pedregal and Tych (1999), Velasco (1999), Phillips (1999), Den Haan and Sumner (2004)). This fact represents a nuisance to the routinely employed interpretation of the consequences of applying linear filters that render the series stationary. Specifically, if we define the spectrum as the limit of the expected periodogram, the spectrum of the (stationary) filtered series is no longer the transfer function of the filter multiplied by the spectrum of the
original (integrated of order 1) series. This conclusion does not hold if the order of integration is only $1 + d$, with $-0.5 < d < 0$.

2 Result

Define $I_{T,x}(\omega_j) = \left| (2\pi T)^{-\frac{1}{2}} \sum_{t=1}^{T} x_t e^{i\omega_j t} \right|^2$ as the periodogram of the sequence $\{x_t\}_{t=1}^{T}$, where $\omega_j = 2\pi j / T$ are the integer multiples of $2\pi / T$ that fall in the interval $[-\pi, \pi]$. Restricting ourselves to real sequences and noting that $I_{T,x}(\omega_j) = I_{T,x}(-\omega_j)$ in this case, we extend as usual the periodogram for every frequency in the interval $[-\pi, \pi]$ in the following way:

$$I_{T,x}(\omega) = \begin{cases} I_{T,x}(g(T, \omega)), & 0 \leq \omega \leq \pi \\ I_{T,x}(-\omega), & -\pi \leq \omega < 0 \end{cases},$$

(1)

where $g(T, \omega)$ is the multiple of $2\pi / T$ closest to $\omega$. If $\{x_t\}_{t=1}^{T}$ is a sample from a stationary time series with mean $\mu$ and the autocovariance function $\gamma_x(\cdot)$ is absolutely summable, one can show (see, e.g., Brockwell and Davis 1991, p.343) that:

$$E[I_{T,x}(0) - (2\pi)^{-1} T \mu^2] \to S_x(0) \text{ as } T \to \infty.$$

(2)

$$E[I_{T,x}(\omega)] \to S_x(\omega) \text{ as } T \to \infty, \; \omega \neq 0.$$

where $S_x(\omega) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} \gamma_x(j) e^{-i\omega j}$ is the spectrum of $\{x_t\}$. That is, when the sample size grows the expectation of the periodogram converges to the distribution of variance as revealed by the spectral representation Theorem. As Solo (1992), in the analysis of the spectrum of continuous-time, stationary increments processes, the question that we address is whether a (power distribution) time-invariance result as in (2) remains valid in the case of integrated processes. The answer is that it does, at least if the order of integration is 1 and for a very broad class of stationary increments. This is summarised in Theorem 1.
**Theorem 1.** Consider the process \( \{x_t\} \) verifying \( x_t - x_{t-1} = u_t = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j} \), where
\[
\sum_{j=-\infty}^{\infty} |\psi_j|^\alpha < \infty \quad \text{for some } \alpha > 0 \quad \text{and } \{\varepsilon_t\} \text{ is a white noise sequence such that } E[\varepsilon_t] = 0 \quad \text{and } \Var[\varepsilon_t] = \sigma^2 < \infty , \forall t. \]
Denote by \( S_{\Delta x}(0) \) the spectrum of \( \{x_t - x_{t-1}\} = \{u_t\} \) at zero frequency and define
\[
S_x^*(\omega) = (2\pi)^{-1} \sigma^2 |\psi(e^{-i\omega})|^2 + |\psi(1)|^2, \omega \neq 0,
\]
where \( \psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j \). Then, the following holds for \( I_{T,x}(\omega) \), the periodogram of \( \{x_t\}_{t=1}^T \):
\[
i) \quad E[T^{-2}I_{T,x}(0)|x_0] \rightarrow \frac{1}{3} S_{\Delta x}(0) \quad \text{as } T \rightarrow \infty.
\]
\[
ii) \quad E[I_{T,x}(\omega)] \rightarrow S_x^*(\omega) \quad \text{as } T \rightarrow \infty, \omega \neq 0.
\]

**Proof:** Consider first the case \( \omega \neq 0 \). Fix any \( \omega \in ]0, \pi] \). Then by (1) \( I_{T,x}(\omega) = I_{T,x}(\omega_j) \) for some Fourier frequency \( \omega_j \). The discrete Fourier transform of \( \{x_t - x_{t-1}\}_{t=1}^T = \{\Delta x_t\}_{t=1}^T \), denoted by \( J_{T,\Delta x}(\omega_j) \), can be decomposed in the following way (as in Phillips 1999, but not fixing \( x_0 = 0 \)):
\[
J_{T,\Delta x}(\omega_j) := (2\pi T)^{-\frac{1}{2}} \sum_{t=1}^{T} \Delta x_t e^{-i\omega_j t} =
\]
\[
= (2\pi T)^{-\frac{1}{2}} (1 - e^{-i\omega_j}) \sum_{t=1}^{T} x_t e^{-i\omega_j t} + (2\pi T)^{-\frac{1}{2}} \sum_{t=1}^{T} x_t e^{-i\omega_j(t+1)} - (2\pi T)^{-\frac{1}{2}} \sum_{t=1}^{T} x_{t-1} e^{-i\omega_j t} =
\]
\[
= (1 - e^{-i\omega_j}) J_{T,x}(\omega_j) + (2\pi T)^{-\frac{1}{2}} (x_T e^{-i\omega_j(T+1)} - x_0 e^{-i\omega_j}). \tag{4}
\]
where \( J_{T,x}(\omega_j) \) denotes the discrete Fourier transform of \( \{x_t\}_{t=1}^T \). Now, the periodogram of \( \{\Delta x_t\}_{t=1}^T \) can be written as \( I_{T,\Delta x}(\omega_j) = J_{T,\Delta x}(\omega_j) J_{T,\Delta x}(-\omega_j) \) (and equivalently for \( I_{T,x}(\omega_j) \)). Multiplying both sides of (4) by \( J_{T,\Delta x}(-\omega_j) \), using the fact that \( e^{-i\omega_j(T+1)} = e^{-i\omega_j} \) for the
Fourier frequencies $\omega_j$ and rearranging terms we get:

$$
|1 - e^{-i\omega_j}|^2 I_{T,x}(\omega_j) = I_{T,\Delta x}(\omega_j) + (2\pi T)^{-1}(x_T - x_0)^2 -
$$

$$
-J_{T,\Delta x}(-\omega_j) (2\pi T)^{-\frac{1}{2}} (x_T - x_0) e^{-i\omega_j} - J_{T,\Delta x}(\omega_j) (2\pi T)^{-\frac{1}{2}} (x_T - x_0) e^{i\omega_j}.
$$

(5)

Now put $R_T(\omega_j) = J_{T,\Delta x}(\omega_j) (2\pi T)^{-\frac{1}{2}} (x_T - x_0) e^{i\omega_j}$ and take expectations:

$$
E[R_T(\omega_j)] = (2\pi T)^{-1} e^{i\omega_j} T \sum_{t=1}^T u_t e^{-i\omega_j t} \sum_{t=1}^T u_t = (2\pi T)^{-1} e^{i\omega_j} 1'E[\mathbf{uu}']e,
$$

where $1$ is a vector of ones, $e = (e^{-i\omega_j}, e^{-2i\omega_j}, ... , e^{-T\omega_j})'$ and $u = (u_1, u_2, ..., u_T)'.$ Since $E[\mathbf{uu}'] = [\gamma_{\Delta x}(j - i)]_{i,j=1}^T$, where $\gamma_{\Delta x}(\cdot)$ is the autocovariance function of $\{\Delta x_t\}$, we get finally:

$$
E[R_T(\omega_j)] = (2\pi T)^{-1} e^{i\omega_j} 1 \sum_{h=1}^T \sum_{l=1}^T \gamma_{\Delta x}(h - l) e^{-i\omega_j h}.
$$

This can be decomposed as follows:

$$
E[R_T(\omega_j)] = (2\pi T)^{-1} e^{i\omega_j} \sum_{h=0}^{T-1} \gamma_{\Delta x}(h) e^{-i\omega_j h} \sum_{l=1}^{T-h} e^{-i\omega_j l} + \sum_{h=-T+1}^{-1} \gamma_{\Delta x}(h) e^{-i\omega_j h} \sum_{l=1-h}^{T} e^{-i\omega_j l}.
$$

(6)

Now, for $0 \leq h \leq T - 1$ we have:

$$
|\sum_{l=1}^{T-h} e^{-i\omega_j l}| = |\sum_{l=1}^{T} e^{-i\omega_j l} - \sum_{l=T-h+1}^{T} e^{-i\omega_j l}| = |0 - \sum_{l=T-h+1}^{T} e^{-i\omega_j l}| \leq h,
$$

since $\sum_{h=1}^{T} e^{-i\omega_j h} = \frac{1-e^{-i\omega_j T}}{1-e^{-i\omega_j}} e^{-i\omega_j} = 0$, as $e^{-i\omega_j T} = 1$ for the Fourier frequencies $\omega_j$. The inequality follows from the fact that $\omega_j \neq 0$ or $2\pi$. Also, for $-T + 1 \leq h \leq -1$ we can conclude that:

$$
|\sum_{l=1-h}^{T} e^{-i\omega_j l}| \leq |h|,
$$
Given (6), all this implies that

\[ |E[R_T(\omega_j)]| \leq (2\pi T)^{-1} \sum_{|h|<T} |\gamma_{\Delta\omega}(h)||h| \leq (2\pi T)^{-1} \sum_{|h|<T} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h| \]

(7)

There are two cases to consider. If \( \alpha < 1 \) then:

\[ (2\pi T)^{-1} \sum_{|h|<T} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h| \leq (2\pi)^{-1} T^{-\alpha} \sum_{|h|<T} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h| \alpha \leq \]

\[ \leq (2\pi)^{-1} T^{-\alpha} \left( \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h + j|^\alpha + \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||j|^\alpha \right) = \]

\[ = 2(2\pi)^{-1} T^{-\alpha} \left( \sum_{h=-\infty}^{\infty} |\psi_h||h|^\alpha \right) \left( \sum_{j=-\infty}^{\infty} |\psi_j| \right) = O(T^{-\alpha}). \]

If \( \alpha \geq 1 \) then:

\[ (2\pi T)^{-1} \sum_{|h|<T} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h| \leq (2\pi T)^{-1} \sum_{|h|<T} \sum_{j=-\infty}^{\infty} |\psi_j\psi_{j+h}||h| \alpha = O(T^{-1}). \]

Performing the same exercise for \( R_T(-\omega_j) \) we conclude that the expected value of the last two terms in (5) converges to 0:

\[ E[R_T(\omega_j) + R_T(-\omega_j)] \rightarrow 0 \text{ as } T \rightarrow \infty. \]  

(8)

As for the second term in (5) we get:

\[ E[(2\pi T)^{-1}(x_T - x_0)^2] = (2\pi T)^{-1} \text{Var} \left[ \sum_{t=1}^{T} \Delta x_t \right] = \]

\[ = (2\pi)^{-1} \sum_{|k|<T} \left( 1 - \frac{|k|}{T} \right) \gamma_{\Delta\omega}(k) \rightarrow (2\pi)^{-1} \sum_{k=-\infty}^{\infty} \gamma_{\Delta\omega}(k) \equiv S_{\Delta\omega}(0) \text{ as } T \rightarrow \infty. \]  

(9)

since \( \gamma_{\Delta\omega}(\cdot) \) is absolutely summable (given that \( \sum_{j=-\infty}^{\infty} |\psi_j| < \infty \)). Using (8), (9), the fact that \( I_{T,\Delta\omega}(\omega) \rightarrow S_{\Delta\omega}(\omega) \) as \( T \rightarrow \infty \) (by (2)) and finally the fact that \( |1 - e^{-i(T,\omega)}|^2 \rightarrow |1 - e^{-i\omega}|^2 \)
as $T \to \infty$ (since $g(T, \omega) \to \omega$) we conclude that:

$$E[I_{T,x}(\omega)] \to \frac{S_{\Delta x}(\omega) + S_{\Delta x}(0)}{|1 - e^{-i\omega}|^2} = (2\pi)^{-1} \sigma^2 |\psi(e^{-i\omega})|^2 + |\psi(1)|^2 |1 - e^{-i\omega}|^2,$$

as $T \to \infty$, $\omega \neq 0$.

For $\omega = 0$, we need to normalise the periodogram by $T^3$ instead of $T$ while fixing $x_0$. We get:

$$E[T^{-2}I_{T,x}(0)|x_0] = E[(2\pi)^{-1}T^{-3} \sum_{t=1}^{T} x_t \sum_{t=1}^{T} x_t |x_0] =$$

$$= E[(2\pi)^{-1}T^{-3} \sum_{t=1}^{T} \sum_{t=1}^{T} u_t + Tx_0)(\sum_{t=1}^{T} \sum_{t=1}^{T} u_t + Tx_0)|x_0] =$$

$$= (2\pi)^{-1}T^{-3} \mathbf{1}' E[u_c u'_c |x_0] \mathbf{1} + (2\pi)^{-1}T^{-1}x_0^2 =$$

$$= (2\pi)^{-1}T^{-3} \mathbf{1}' E[u_c u'_c] \mathbf{1} + O(T^{-1}),$$

where $\mathbf{1}$ is a vector of ones and $u_c = (u_1, u_1 + u_2, ..., \sum_{t=1}^{T} u_t)$. Evaluating $E[u_c u'_c]$ we conclude that

$$E[T^{-2}I_{T,x}(0)|x_0] = (2\pi)^{-1}T^{-3} \sum_{|k|<T} \gamma_{\Delta x}(k) \sum_{h=1}^{T-|k|} (|k| + h)h + O(T^{-1}).$$

Noting that

$$\sum_{h=1}^{T-|k|} (|k| + h)h = \frac{(T - |k|)(T - |k| + 1)(2(T - |k|) + 1)}{6} +$$

$$\frac{(T - |k|)(T - |k| + 1)|k|}{2} = T^3 R(T, |k|)/3, \text{ say,}$$

we get

$$E[T^{-2}I_{T,x}(0)|x_0] = \frac{(2\pi)^{-1}}{3} \sum_{|k|<T} \gamma_{\Delta x}(k) R(T, |k|) + O(T^{-1}),$$

where, for fixed $|k|$, $\lim_{T \to \infty} R(T, |k|) = 1$. Since $\gamma_{\Delta x}(\cdot)$ is absolutely summable, we finally conclude:

$$E[T^{-2}I_{T,x}(0)|x_0] \to \frac{1}{3} S_{\Delta x}(0)$$

as $T \to \infty$. 

\[\blacksquare\]
Example: Random walk. If \( \{x_t\} \) verifies \( x_t - x_{t-1} = \varepsilon_t, \forall t \) where \( \{\varepsilon_t\} \) is a white noise sequence such that \( E[\varepsilon_t] = 0 \) and \( Var[\varepsilon_t] = \sigma^2 \) we have, since \( \psi(e^{-i\omega}) = \psi(1) = 1 : \)

\[
S_x^*(\omega) = \frac{\sigma^2}{\pi|1 - e^{-i\omega}|^2}, \omega \neq 0,
\]

which shows that the pseudo-spectrum \( S_x^*(\omega) \), defined as in Theorem 1, is just proportional to the inverse of the Fourier transform of the differencing operator \( (1 - L) \), where \( L \) is the lag operator. However, if we apply the first difference filter to \( \{x_t\} \) the spectrum of \( (1 - L)x_t = \varepsilon_t \) is given by \( S_\varepsilon(\omega) = \sigma^2/2\pi \). To perfectly maintain the relation \( S_\varepsilon(\omega) = |1 - e^{-i\omega}|^2 S_x(\omega) \) (as it would be the case if \( \{x_t\} \) were stationary) we would need to define the pseudo-spectrum of \( \{x_t\} \) as:

\[
S_x(\omega) = \frac{\sigma^2}{2\pi|1 - e^{-i\omega}|^2}, \omega \neq 0,
\]

which seems a neutral normalization of \( S_x^*(\omega) \), a (non-integrable) power distribution of \( \{x_t\} \). In this case the first difference filter maintains the usual interpretation, summarised by the function \( |1 - e^{-i\omega}|^2 \). It attenuates low frequencies and amplifies high frequencies, thus producing a “noisier” output series. ■

Remark 1. Theorem 1 covers long memory increments for which \( u_t = \psi(L)\varepsilon_t = (1 - L)^{-d}\psi^*(L)\varepsilon_t \) where \( \{\psi^*(L)\varepsilon_t\} \) is weakly dependent, \(-0.5 < d < 0 \) and \( (1 - L)^{-d} := \sum_{j=0}^{\infty} \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} L^j \), where \( \Gamma(y) = \int_0^\infty z^{y-1}e^{-z}dz \). We say \( \{x_t\} \) is integrated of order \( 1 + d \). It is easy to verify that

\[
\sum_{j=-\infty}^{\infty} \rho_j |j|^{-d-\varepsilon} < \infty, \text{ for some } 0 < \varepsilon < -d, \text{ since } \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(j+1)} \sim j^{d-1}/\Gamma(d) \text{ as } j \to \infty \text{ and } \{\psi^*(L)\varepsilon_t\} \text{ is weakly dependent (by the same token, Theorem 1 is not applicable if } 0 < d < 0.5 \).

It turns out that \( \{x_t\} \) is non-stationary but Theorem 1 delivers \( S_x^*(\omega) = (2\pi)^{-1}\sigma^2 |\psi^*(e^{-i\omega})|^2 |1 - e^{-i\omega}|^{-2(1+d)}, \omega \neq 0, \) since \( S_{\Delta x}(0) = \psi(1) = 0 \). Thus, application of the first-difference filter to \( x_t \) leads to the spectrum \( S_{\Delta x}(\omega) = (2\pi)^{-1}\sigma^2 |\psi^*(e^{-i\omega})|^2 |1 - e^{-i\omega}|^{-2d}, \omega \neq 0, \) meaning the relation \( S_{\Delta x}(\omega) = |1 - e^{-i\omega}|^2 S_x^*(\omega) \), as in the stationary case, is maintained. ■

Remark 2. A different normalisation is needed for convergence if the order of integration
is greater than 1. Consider the simplest case \((1 - L)^2 x_t = \varepsilon_t, \forall t\) where \(\{\varepsilon_t\}\) is a white noise sequence. With \(x_0 = 0\) and \(\sigma^2 = 1\), it is easy to verify that:

\[
E[I_{T,x}(\omega)] = \sum_{|k|<T} T^{-1} \cos[g(T, \omega)k] \sum_{t=1}^{T-|k|} t(|k| + t), \omega \neq 0
\]

which diverges since \(\sum_{t=1}^{T-|k|} t(|k| + t)\) is a polynomial of order 3 in \(T\) and when \((1 - L)x_t = \varepsilon_t\),

\[
E[I_{T,x}(\omega)] = \sum_{|k|<T} T^{-1} \cos[g(T, \omega)k](T - |k|)(T - |k| + 1), \omega \neq 0
\]

is convergent (by Theorem 1).

We shall not pursue any frequency domain characterisation in this case. ■

Theorem 1 is an adaptation of the continuous-time result in Solo (1992). Also, it sharpens the result of Theorem 4 in Crato (1996) which gives an upper bound greater than 0 to the limit of (6). Part i) is nonetheless covered in Lemma 4.1 of Robinson and Marinucci (2001). In a fractional integration context including unit roots, Hurvich and Ray (1995) have studied the behaviour of the expectation of the periodogram at Fourier frequencies close to the origin, obtaining also a time-invariance result. Specifically, Theorem 1 in Hurvich and Ray (1995) shows the following, for a unit-root process:

\[
E[I_{T,x}(\omega_j)/S_x(\omega_j)] \rightarrow 2 \text{ as } T \rightarrow \infty, \omega_j = 2\pi j/T.
\]  

(10)

where

\[
S_x(\omega) = \frac{\sigma^2}{2\pi} \left| \psi(e^{-i\omega}) \right|^2, \omega \neq 0,
\]  

(11)

\(S_x(\omega)\), which differs from \(S^*_x(\omega)\) in Theorem 1 (see discussion below), is interpreted as the spectrum of the integrated series as is in Velasco (1999) and Phillips (1999). Phillips (1999) argues that \(S_x(\omega)\) has such interpretation in view of Solo’s (1992) argument (i.e., \(S_x(\omega)\) would be the limit of the expectation of the periodogram, which is not exactly true). It should be noted that \(j\) is held fixed, whereas our result is valid for any fixed \(\omega \neq 0\). It is easy to reconcile the two results. Heuristically, once \(T\) grows, \(\omega_j\) approaches 0 and hence \(\left| \psi(e^{-i\omega}) \right|^2\) approaches \(\left| \psi(1) \right|^2\). Therefore \(\frac{1}{2\pi} I_{T,x}(\omega_j)\) approaches \(2S_x(\omega_j)\). In the stationary case the limit in (10) is
just 1 whereas for processes with order of integration $1 + d$ ($-0.5 < d < 0$), the limit depends on $j$.

3 Interpreting filtered integrated time series

If we apply to the stationary sequence $\{x_t\}$ a time-invariant linear filter $h(L) = \sum_{j=-\infty}^{\infty} h_j L^j$, such that $\sum_{j=-\infty}^{\infty} |h_j| < \infty$ we obtain a filtered sequence $y_t = \sum_{k=-\infty}^{\infty} h_j x_{t-j}$. The spectrum of $\{x_t\}$, $S_x(\omega)$, is related to that of $\{y_t\}$, $S_y(\omega)$, by

$$S_y(\omega) = |h(e^{-i\omega})|^2 S_x(\omega).$$

(12)

Now, if we define the spectrum of an integrated process as the limit of the spectrum of a stationary process when the smallest autoregressive roots converge to 1 (e.g., Harvey (1993), Den Haan and Sumner (2004), Young, Pedregal and Tych (1999)) we will get, for a general ARIMA($p, 1 + d, q$), $-0.5 < d \leq 0$ process, a pseudo-spectrum defined as:

$$S_x(\omega) = \frac{\sigma^2}{2\pi} \frac{\phi^{-1}(e^{-i\omega})^2 |\theta(e^{-i\omega})|^2}{|1 - e^{-i\omega}|^2(1+d)} = \frac{\sigma^2}{2\pi} \frac{\psi(e^{-i\omega})^2}{|1 - e^{-i\omega}|^2}, \omega \neq 0,$$

(13)

where $x_t$ satisfies:

$$\phi(L)(1 - L)x_t = (1 - L)^{-d}\theta(L)\varepsilon_t, \ \forall t.$$

$\sigma^2$ is the variance of the white-noise innovations $\varepsilon_t$, we assume the roots of $\phi(L)$ lie outside the unit circle and are different from those of $\theta(L)$ and $\psi(L) = \phi(L)^{-1}(1 - L)^{-d}\theta(L)$. This limit is a time-invariant function but it is not well defined at those frequencies associated with autoregressive roots with unit modulus\(^1\). Further, it equals $S_x(\omega)$ in (11). An extension of the relation in (12) holds obviously given the definition in (13), particularly when the filter renders the series stationary. E.g., consider the case of the ideal band-pass filter $h(L)$, where

\(^1\)Since we assumed the roots of $\phi(L)$ lie outside the unit circle, we are only considering the existence of a pole at zero frequency. This assumption can straightforwardly be relaxed in order to include singularities at frequencies other than zero, e.g., due to non-stationary seasonal components.
\[ |h(e^{-i\omega})| = 1 \] for a range of frequencies \([\omega_l, \omega_h] \subset [0, \pi] \) and 0 otherwise. The filter weights \( \{h_j\} \) are well-known and given by:

\[
h_o = \frac{\omega_h - \omega_l}{\pi}, \quad h_j = \frac{\sin[\omega_hj] - \sin[\omega_lj]}{\pi j}, |j| \geq 1 \tag{14}
\]

Since \( \{y_t\} \equiv \{h(L)x_t\} \) is stationary (since \( h(1) = 0 \)), it is obvious that \( S_y(\omega) = S_x(\omega) \) for the range of frequencies \([\omega_l, \omega_h]\) and 0 otherwise. Further, \( S_y(0) = 0 \) since \( h(L) \) has not one but two unit-roots. We thus know for sure that no fluctuations with frequencies outside the band of interest remain in the filtered series when \( \{x_t\} \) is integrated of order 1 + \( d \) (\(-0.5 < d \leq 0\)), even if \( S_x(\omega) \) is regarded as meaningless. That is, although the interpretation of band-pass (or other) filters implicit in (12) is routinely (and in our view, sensibly) employed, it is assumed, without resorting to results such as that in Theorem 1, that this function represents indeed a (nonetheless degenerate) distribution of variance.

The unfortunate fact is that the time-invariant limit in Theorem 1 in the canonical case of \( d = 0 \) (the usual assumption for macroeconomic time series) is slightly different than that in (13) due to the term \(|\psi(1)|^2\) in the numerator. This is definitely a nuisance when the process is not a pure random walk, for which a straightforward normalisation (as in Example 1) preserves the (degenerate) power distribution and leads to the maintenance of the relation in (12). In any case, and given this normalisation, the differences in the interpretation would not be dramatic given the fact that the inverse of \(|1 - e^{-i\omega}|^2\) dominates the behaviour of both functions at frequencies close to the pole located at zero frequency (see also the result in (10)).

In short, one could also be tempted to define the limit of the expected periodogram as the spectrum (or degenerate power distribution) of the integrated (of order 1) process, since in the stationary case this limit is the distribution of power revealed by the spectral representation Theorem. However, if the order of integration equals 1, this function differs from the commonly defined (pseudo-) spectrum of an integrated time series. Defining the spectrum of an integrated (of order 1) series as in Theorem 1 would in general distort the interpretation given to the transfer function of filters applied to such series.
References


