A projection of the market based adjustment path of the Portuguese Economy

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Abstract

The Portuguese Economy entered the Global Financial Crisis with large external macroeconomic imbalances. In this work I estimate the speed of the market mechanism at work when a country adjusts to external imbalances within a currency area known as competitive disinflation or internal devaluation. I provide benchmark projections for the unemployment rate, for the final prices inflation, for the labor compensation inflation and for the real exchange rate dynamics. The scope of the exercise is to provide an operational guide to the policy maker without necessarily taking a stance on the nature of the forecasted variables: goals, intermediate targets or indicators. The results indicate that such a mechanism is indeed at work in the data and that it is slow. Starting the projection from the fourth quarter of 2011, I project after 20 quarters an increase of the unemployment rate between

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4 and 6 percentage points and an improvement of the real exchange rate of approximately 5 percentage points. Obviously shocks will happen and policies will be taken and the actual path will be different. Nevertheless the forecasts can be interpreted as a counterfactual path in the absence of new shocks and policies and therefore should be of some guidance in designing policies aimed at rebalancing the external adjustment. Since the various methodological aspects I combine are known, this paper might be regarded as essentially expository.
**Introduction**

The Portuguese Economy entered the Global Financial Crisis with large external macroeconomic imbalances. The objective of this work is to estimate the market-based trajectory for the Portuguese Economy induced by those imbalances. The estimated trajectory is not a prediction but a benchmark path that we should expect in the absence of shocks and policies. In order to perform this task, I need to estimate the postulated macroeconomic mechanism at work when a country adjusts to external imbalances within a currency area. This mechanism is known as competitive disinflation, or internal devaluation, and is based on the downward pressure exerted by unemployment on wages and prices which in turn restore competitiveness. In this work I estimate the speed of the market mechanism and provide benchmark projections for the unemployment rate, for the final prices inflation, for the labor compensation inflation and for the real exchange rate dynamics. The scope of the exercise is to provide an operational guide to the policy maker without necessarily taking a stance on the nature of the forecasted variables: goals, intermediate targets or indicators.

**Portugal competitiveness**

I present the recent history of Portugal external imbalances in Figure 1 (see Blanchard 2007 for a more exhaustive discussion). The narrative of the imbalances is as follow: from 1995 to 2001 the large decrease in nominal interest rate\(^1\) (panel 1) fueled an expansion in private expenditure (panel 2) financed with debt (panel 3). The increase in

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\(^1\)The decrease in interest rates can be considered exogenous as it was driven by convergence towards German rates. The same convergence happened across all the peripheral countries of the euro-area. In my undergraduate thesis I had raised the question on the fundamentals of the convergence towards German rates as opposed to an average of the rates of the future euro participants.
demand pushed nominal labor compensation to ran a rate of 6 percent per annum, a rate well above labor productivity, and GDP inflation to increase to 4 percent per annum. The result was a large and rapid loss in competitiveness vis-a-vis the eurozone partners (panel 4). During and after the recession of 2002, labor compensation and final prices inflation decelerated, but not sufficiently, and Portugal competitiveness continued to, this time slowly, deteriorate against the other eurozone members (panel 4). The first panel of Figure 2 shows that a measure of REER based on ULC vis-a-vis the rest of the eurozone, normalized to 100 in 1995, was equal to 83.4 in the the fourth quarter 2011 implying a cumulative loss of 16.6 percentage points. A similar measure based on GDP deflators was at 88.3 implying a cumulative loss of 11.8 percentage points.

At the onset of the global financial crisis Portugal external imbalances were the largest of the eurozone members (-110 percent of GDP in the net international investment position) and most of the current account deficit was financed and intermediated by the other eurozone members banks. When the crisis hit the eurozone, financial markets questioned the sustainability of the external position, interest rates soared and the eurozone central banks had to substitute commercial banks in financing the current account. While the fragility of the eurozone financial system has to be solved at the currency area level, the rebalancing of the current account, in particular through an improvement in competitiveness to shift to a positive trade balance, needs to be addressed at the national level. The external rebalancing is difficult as the degrees of freedom of the Portuguese authorities are limited in number: they have no autonomous monetary policy, no currency to devaluate, and after having accepted the conditionality to access external credit lines by the Trojka, no discretion on fiscal policy. Without economic policies the adjustment must logically come through the self-equilibrating forces of the market\(^2\). The

\(^2\)In a previous work, I had taken for given that internal devaluation was a slow process, most probably
market based mechanism that takes place within a currency area or a fixed exchange rate system to adjust to external imbalances (in the absence of fiscal transfers and with limited labor mobility/migration) is called competitive disinflation or internal devaluation and can be described piece-wise as follows:

- the loss of competitiveness and the deterioration of the trade balance leads to an increase in unemployment,
- the increase in unemployment pushes the growth rate of nominal wages down,
- the decrease in wage inflation is transmitted to final prices inflation,
- the decrease in final prices inflation lead to an improvement in competitiveness,
- the improvement in competitiveness rebalances the trade balance and the current account,
- the increase in exports decreases the unemployment rate.

The rest of the paper is concerned with estimation of this mechanism. The next section presents the methodological framework I follow to perform the proposed exercise.

**Methodology**

During the last three decades Dynamic Stochastic General Equilibrium models have emerged as the dominant framework in macroeconomics. The theoretical advantages of because of substantial downward nominal rigidities in wages and market power in the non-tradable sector, and suggested a fiscal devaluation as a policy to accelerate the adjustment. Political constraints and uncertainty on the quantitative efficacy of the fiscal devaluation have pushed Portugal to follow a path, for what regards the short to medium run competitiveness, mostly determined by the self-healing forces of the market.
the DSGE methodology that help explain its success are based in the specification of macroeconomic models derived from microeconomic principles. The specification of a model with “clear foundations in individual optimization” Woodford (2003) implies that DSGE models, unlike more traditional macro-econometric forecasting models, are not subject to the Lucas critique and that policy experiments can be evaluated in terms of a well specified Welfare criterion. Finally during the last decade the tremendous improvement in computational tools available has allowed macroeconomists to estimate DSGE models. Still after all the tremendous efforts to develop the framework, DSGE are thought not to be useful for policy analysis (Chari et al 2008) and perform somehow poorly in forecasting out of sample (Edge and Gurkaynak 2011). Typically a DSGE study starts by laying down the details of the economy, found a solution and finally estimate the solution. There are different numerical approaches to solve and estimate DSGE. The log-linear approximation around a steady state is the most popular one. Log-linear approximations calculated by using the optimality conditions of the problem, have proven extremely fruitful in taking DGSE models to the data because the solution takes the form of linear state space model whose likelihood can be maximized using the Kalman filter. I suggest to take stock of the DSGE methodology and complement its inductive nature with a more practical empirical approach in order to improve specification and quantitative performance.

The empirical approach I follow starts from a generic class of DSGE where by class of DSGE I intend all the particular specifications that lead to the same linear state space model\(^3\). Within that particular class of DSGE, I identify the structural fundamental shocks that explain the data. This is reminiscent of the Structural Vector Autore-

\(^3\)The fact that many rational expectations models lead to the same reduced form has been recognized long ago and is called observational equivalence.
gressions (SVAR) approach, a data based method to study macroeconomic fluctuations which to begin with was the competitor and subsequently became a complementary tool to the DSGE framework. Working directly with the state space representation, as opposed to a SVAR, has in my opinion several advantages. First it is methodologically coherent with the subsequent step of estimating the structural parameters (which is not attempted here). Second it allows to study the large class of structural models that can be casted in a VARMA. Third it allows to easily map the parameters and cross-equation restrictions implied by the DSGE and check if the shocks are identified.

To summarize, the first step of the methodology is to present a state of the art micro-founded model (a New Keynesian model of a Small Open Economy within a currency union). However in the empirical part, the second step, I do not estimate the “structural” parameters of the micro-founded model but the parameters of the implied statistical model. The first step is therefore present to have a coherent and disciplined example to interpret the data. The second step estimates a model consistent with the first step but also with many other model adding an healthy dose of pragmatism to the exercise. One cost of this approach is that the estimates cannot be used for the evaluation of policies. A benefit of this approach is that the implied statistical model is consistent with different micro-foundations (for example consistent with different stories on why prices and wages are sticky). The empirical exercise focuses on the global identification of the reduced form elasticities and of the structural shocks of a class of DSGE as opposed to the local identification of the structural parameters of a particular DSGE. The structural

\[\text{\footnotesize 4Indeed SVAR analysis has been used to provide limited information on the properties of the DSGE. For instance Christiano and Eichenbaum identify a monetary policy shocks from SVAR and use the information of the response of the economy to that shock both to specify and estimate the parameters of the DSGE.}\]

\[\text{\footnotesize 5Another potential advantage is that the process of filtering to obtain cyclical information from the actual data can be made more consistent and transparent by performing it directly on suitably specified state space model.}\]
shocks being exogenous non-modeled forces such as policy actions, supply shocks and
demand shocks\(^6\).

**Step I: a model**

I use the framework by Farhi and Werning (2012) who develop an incomplete financial
markets version of the Gali and Monacelli (2005) model of a small open economy. To
their framework I add nominal wage stickiness as in Erceg, Henderson and Levin (2000)
and unemployment as in Gali (2011) in order to derive reduced form consistent with the
empirical exercise.

**Countries**

There is a continuum of countries indexed by \(i \in [0, 1]\) of measure one. The small open
economy is called Home and can be thought as a particular value \(H \in [0, 1]\) that takes
as given the rest of the world which is called Foreign.

**Households**

Home has a continuum of households indexed by a pair \((h, l) \in [0, 1] \times [0, 1]\). The first
dimension, indexed by \(h\), represents the type of labor service in which a given household
member is specialized. The second dimension, indexed by \(l\), determines his disutility
from work. The latter is given by \(\chi l^{\phi}\) if he is employed and zero otherwise, where
\(\phi\) determines the elasticity of the marginal disutility of work, and \(\chi\) is an exogenous

\(^6\)Notice that if interested in Welfare analysis, the identification of the structural shocks is also neces-
sary when the estimation is done on the structural parameters of a particular DSGE.
preference parameter. The household seeks to maximize the utility of all of its members (there is full risk sharing among household members) given by

\[
\sum_{s=0}^{\infty} \beta^s \left[ \frac{C_{t+s}^{-1} - \frac{1}{\sigma}}{1 - \frac{1}{\sigma}} - \chi \int_0^1 N_{t+s}^{1+\phi}(h) \, dh \right],
\]

where \( N_t(h) \) is the quantity of labor of type \( h \) supplied. Workers supplying a labor service of a given type have some monopoly power in the labor market, and post the (nominal) wage at which they are willing to supply specialized labor services to firms that demand them. Assume that for each period only a fraction \( 1 - \delta_w \) of labor-type worker, drawn randomly from the population, re-optimize their posted nominal wage. Under the assumption of full consumption risk sharing across households, all households resetting their wage in any given period will choose the same wage, because they face an identical problem. \( C_t \) is a composite consumption index define by

\[
C_t = \left[ (1 - \alpha)^{\frac{1}{\pi}} C_{H,t}^{\frac{\pi - 1}{\pi}} + \alpha \frac{1}{\pi} C_{F,t}^{\frac{\pi - 1}{\pi}} \right]^{\frac{\eta}{\eta - 1}},
\]

where \( C_{H,t} \) is an index of consumption of domestic goods given by a constant elasticity of substitution aggregator

\[
C_{H,t} = \left[ \int_0^1 (C_{H,t}(j))^{\frac{\pi - 1}{\pi}} \, dj \right]^{\frac{\pi}{\pi - 1}},
\]

where \( j \in [0,1] \) denotes an individual good variety. Similarly \( C_{F,t} \) is a consumption index of imported goods given by

\[
C_{F,t} = \left[ \int_0^1 (C_{i,t})^{\frac{\pi - 1}{\pi}} \, di \right]^{\frac{\pi}{\pi - 1}},
\]

9
where $C_{i,t}$ is in turn, an index of the consumption of varieties of goods imported from country $i$, given by

$$C_i^t = \left[ \int_0^t \left( C_i^t(j) \right)^{\frac{\epsilon-1}{\epsilon}} dj \right]^{\frac{\epsilon}{\epsilon-1}}.$$ 

Thus, $\epsilon$ is the elasticity of substitution between varieties produced within a given country, $\eta$ is the elasticity between domestic and foreign goods, and $\gamma$ is the elasticity between goods produced in different foreign countries. The parameter $\alpha$ indexes the degree of home bias, and can be interpreted as a measure of openness. Maximization of utility is subject to a sequence of budget constraints of the form

$$(1 + \tau_{c,t}) \left( \int_0^1 P_{H,t}(j) C_{H,t}(j) dj + \int_0^1 \int_0^1 P_{i,t}(j) C_{i,t}(j) djdj \right)$$

$$+ B_{t+1} + \int_0^1 B_{i,t+1} di \leq R_{t-1} B_t + \int_0^1 R_{i,t-1} B_{i,t+1} di + \int_0^1 W_t(h) N_t(h) dh + \Pi_t + T_t$$

where $\tau_{c,t}$ is a proportional tax on consumption, $P_{H,t}(j)$ is the price of domestic variety $j$, $P_{i,t}$ is the price of variety $j$ imported from country $i$, $W_t$ is the nominal wage, $\Pi_t$ represents nominal profits and $T_t$ is a nominal lump sum transfer. $B_{t+1}$ is a domestic nominal bond, $R_t$ is the nominal gross interest rate on the nominal bond, $B_{i,t+1}$ is bond holding of country $i$ of home household nominal gross interest rate equal to $R_{i,t}$. The optimal allocation of any given expenditure within each category of goods yields the demand functions

$$C_{H,t}(j) = \left( \frac{P_{H,t}(j)}{P_{H,t}} \right)^{-\epsilon} C_{H,t},$$

$$C_{i,t} = \left( \frac{P_{i,t}}{P_{F,t}} \right)^{-\gamma} C_{F,t},$$

$$C_{i,t}(j) = \left( \frac{P_{i,t}(j)}{P_{i,t}} \right)^{-\epsilon} C_{i,t}.$$
\[ P_{H,t} = \left[ \int_0^1 P_{H,t}(j)^{1-\varepsilon} \,dj \right]^{\frac{1}{1-\varepsilon}}, \quad P_{F,t} = \left[ \int_0^1 P_{F,t}(j)^{1-\gamma} \,dj \right]^{\frac{1}{1-\gamma}}, \quad P_{t,t} = \left[ \int_0^1 P_{t,t}(j)^{1-\varepsilon} \,dj \right]^{\frac{1}{1-\varepsilon}} \]

are the price indexes for each category of goods. The optimal allocation of expenditure in tradable goods between domestic and foreign implies

\[ C_{H,t} = \left( \frac{P_t}{P_{H,t}} \right)^\eta (1-\alpha) C_t, \]
\[ C_{F,t} = \left( \frac{P_t}{P_{F,t}} \right)^\eta \alpha C_t, \]

where \( P_t = \left[ (1-\alpha) (P_{H,t})^{1-\eta} + \alpha (P_{F,t})^{1-\eta} \right]^{\frac{1}{1-\eta}} \) is the consumer price index (CPI). A household resetting its wage \( W_t^r(h) \) in period \( t \), maximizes its utility subject to the budget constraint and the demand for labor services

\[ N_{t+k|t}(h,j) = \left( \frac{W_t^r(h) \omega_{t+k}}{W_{t+k}} \right)^{-\varepsilon_w} N_{t+k}(j) \]

where \( W_t = \left[ \int_0^1 W_t(h)^{1-\varepsilon_w} \,dh \right]^{\frac{1}{1-\varepsilon_w}} \) is an aggregate wage index and \( \omega_{t+k} \) is

\[ \omega_{t+k} = \begin{cases} 1 & k = 0 \\ \prod_{s=0}^{k-1} \Pi_{t+s}^s & k \geq 1 \end{cases} \]

and denotes the degree of wage indexation to past inflation for households that cannot reset optimally the wage\textsuperscript{7}. The marginal supplier of type \( h \) labor (employed or unemployed) which is denoted by \( L_t(h) \), is implicitly given by

\textsuperscript{7}Indexation is present so that the wage inflation rate is a state variable and not a control variable coherently with what observed in the data below. The motivation is purely empirical and illustrates the type of ad-hoc “micro-foundations” that are sometimes embedded in DSGE in order to improve their fit.
\[ W_t(\lambda) = P_t X_t C_t^\sigma L_t(\lambda)^{\phi} (1 + \tau_{c,t}) . \]

**Firms**

The economy is composed of one tradable sector. There is a continuum of firms indexed by \( j \in [0, 1] \), each of which produces a differentiated good with the following technology

\[ Y_t(j) = A_{H,j} N_t(j) \]

where \( Y_t(j) \) denotes the output of good \( j \), \( A_{H,j} \) is an exogenous technology parameter, and \( N_t(j) \) is an index of labor input used by firm \( i \) and defined by

\[ N_t(j) = \left[ \int_0^1 N_t(j,h) \frac{e_w^{-1}}{e_w} dh \right]^{-e_w} , \]

where \( N_t(j,h) \) is the quantity of type-\( h \) labor employed by firm \( i \) in period \( t \). The parameter \( e_w \) represents the elasticity of substitution among labor varieties. The demand schedule for each labor type is obtained by cost minimization

\[ N_t(j,h) = \left( \frac{W_t(h)}{W_t} \right)^{-e_w} N_t(j) , \]

for all \( j,h \in [0, 1] \), where \( W_t(h) \) is the nominal wage for type \( h \) labor and \( W_t = \left[ \int_0^1 W_t(h)^{1-e_w}dh \right]^{-1/e_w} \) is an aggregate wage index. All firms face an iso-elastic demand schedule (specified below). Firms must pay a social contribution in the form of a proportional tax, \( \tau_{w,t} \), on their wage bill so that their real marginal cost deflated by home PPI is given by \( MC_t = \frac{1 + \tau_{w,t}}{A_{H,j}} \frac{W_t}{P_{H,j}} \). Finally each firm may reset its price with probability \( 1 - \delta_p \) in any given period independently of the time elapsed since the last adjustment.
Those firms that get to reset their price choose a reset price $P^r$ to solve

$$
\sum_{k=0}^{\infty} (\delta_p)^k \{ \Psi_{t,t+k} Y_{t+k|t} (j) (P^r_t(j) - P_{H,t+k}(j)MC_{t+k}) \}
$$

where $Y_{t+k|t}(j) = \left( \frac{P^r_t(j) \xi_{t,t+k}}{P_{H,t+k}} \right)^{-\epsilon} Y_{t+k}$, $\Psi_{t,t}$ is the stochastic discount factor and $\xi_{t,t+k}$ is

$$
\xi_{t,t+k} = \begin{cases} 
1 & k = 0 \\
\prod_{s=0}^{k-1} \prod_{H,t+s}^0 & k \geq 1 
\end{cases}
$$

and denotes the degree of price indexation to past inflation.

### International Prices

The real exchange rate is

$$
Q_t = \frac{P^*_t}{P_t},
$$

where $P^*_t = \left[ \int_0^1 P_{i,t}^{1-\gamma} di \right]^{1-\gamma}$ and the terms of trade is

$$
S_t = \frac{P_{F,t}}{P_{H,t}} = \left( \int_0^1 S_{i,t}^{1-\gamma} di \right)^{1-\gamma},
$$

where $S_{i,t} = P_{i,t}/P_{H,t}$.

### Aggregate Conditions

Given the assumed wage setting structure, the evolution of the aggregate wage index is given by

$$
W_t = \left[ \delta_w W_{t-1}^{1-\epsilon_w} + \left( 1 - \delta_w \right)(W_t^r)^{1-\epsilon_w} \right]^{1/1-\epsilon_w},
$$
while the assumed price setting structure implies

\[ P_t = \left[ \delta_p P_{t-1}^{1-\epsilon} + (1 - \delta_p) (P_t^*)^{1-\epsilon} \right]^{\frac{1}{1-\epsilon}}. \]

Market clearing in the tradable goods market requires that

\[ Y_{H,t}(j) = C_{H,t}(j) + \int_0^1 C^i_{H,t}(j) di, \]

holds in every period. Letting aggregate output be defined as

\[ Y_{H,t} \equiv \left( \int_0^1 Y_{H,t}(j) \frac{\epsilon - 1}{\epsilon} dj \right)^{\frac{\epsilon}{\epsilon - 1}}, \]

it follows that

\[ Y_{H,t} = \left( \frac{P_{H,t}}{P_t} \right)^{-\eta} \left[ (1 - \alpha) C_t + \alpha \int_0^1 (Q^j_t)^{\eta} (S_t S^j_t)^{\gamma - \eta} C^j_t di \right]. \]

Market clearing in the labor market requires

\[ N_t = \frac{Y_t}{A_{H,t}} \int_0^1 \left( \frac{P_t(j)}{P_t} \right)^{-\epsilon} dj. \]

The aggregate labor force is given by

\[ L_t = \left( (1 + \tau_{c,t})^{-1} \chi^{-1} \frac{W_t}{P_t} C_t^{-\sigma} \right)^{\frac{1}{\delta}} \int_0^1 \left( \frac{W_t(\lambda)}{W_t} \right) \frac{1}{\delta} d\lambda, \]

and unemployment is

\[ U_t = L_t - N_t. \]
Fiscal Policy

The fiscal authority is assumed to rebate tax income to households

\[ \tau_{c,t}C_t + \tau_{w,t}W_tN_t = T_t = \int_0^1 T_t(h)dh. \]

Monetary policy

The central bank runs a common monetary policy for all countries, responding only to aggregate union-wide variables (U) that is represented by a Taylor Rule

\[ R_t^* = \bar{R} \left( \frac{P_t^U}{P_t^{U,t-1}} \right)^{\phi_p}, \]

where \( \phi_p \) denotes the feedback coefficient associated with the union wide inflation gap (where the target is assumed to be zero). Given the focus on the Home economy, the policy interest rate is exogenous. Finally the nominal interest rate equals the policy rate plus a risk premium related to the net external position \( NFA \):

\[ R_t = \bar{R} + \psi \left( e^{\frac{NFA_t}{\bar{R}}} - 1 \right). \]

Shocks

To provide the mapping between the theoretical model and the empirical exercise presented below, I characterize the response of the model economy to various one-time unanticipated shocks (a perfect foresight exercise). I assume that the economy is initially at the deterministic symmetric steady state and consider shocks to productivity \( \{A_{H,t}\}_{t \geq 0} \), shocks to the policy rate \( \{R_t\}_{t \geq 0} \), shocks to the labor supply \( \{\ell_t\}_{t \geq 0} \) and
shocks foreign consumption \( \{C_t^e\}_{t \geq 0} \). I focus on the response of the four variables that I use in the estimation part, namely \( \Pi, \Pi_p, U/L \) and \( S \). Figures 3-6 show the response of the four variables of interest to each shock.

Consider the labor productivity shock in Figure 3. The shock has a deflationary effect on final prices and wages and increases both unemployment and the terms of trade. Label this shock a supply shock. Figure 4 shows the IRFS to the policy rate shock (a decrease in the rate). The shock is inflationary and expansionary as it decreases unemployment and improve the terms of trade. Figure 5 and 6 show the IRFS to the labor supply (a decrease in \( \chi \)) and to foreign consumption (an increase). The effects are qualitatively similar to the cut in the rate except for the initial response of inflation in the case of the labor supply shock. Label these three shocks as demand shocks.

While the exercise is performed using standard values for the parameters, the shape and sometimes the sign of the response depend on the values of those same parameters. In this configuration of the parameters, the supply shock commands a positive comovement between unemployment and the terms of trade while the three demand shocks command a negative comovement between the same two variables. My limited discussion on the characterization of the shocks reflects the illustrative scope of the model but still permits a disciplined interpretation of the empirical results below.

The empirical strategy

The empirical approach starts with two simplifications of the DSGE model presented above. First, I work with a log-linearized version of the model which permits to use linear econometrics and facilitates the interpretation of the reduced form parameters as elasticities. Second I focus on the supply of the model and reduce the demand bloc to
one equation in order to reduce the number of parameters to be estimated by focusing only on the elasticities relevant for the internal devaluation mechanism. In practice I need to further reduce the supply side of the model because of the small number of observations for the Portuguese economy.

The log-linear model

I start by presenting the log-linear model and the dynamics implied by different type of shocks. The supply block of the DSGE model can be approximated around the steady state by wage-Phillips Curve an output price-Phillips Curve and the definition of the CPI (see Appendix):

\[
\pi_t^w - \gamma_w \pi_{t-1}^c = \beta \left( \pi_{t+1}^w - \gamma_w \pi_t^c \right) - \kappa_w (u_t - u_t^n) + z_t^w,
\]

\[
\pi_t^p - \gamma_p \pi_{t-1}^p = \beta \left( \pi_{t+1}^p - \gamma_p \pi_t^p \right) + \kappa_p \left( (\pi_t^w - \pi_t^c) + (1 - \alpha) \left( \pi_t^p - \pi_t^p \right) \right) + z_t^p,
\]

\[
\pi_t^c = \alpha \pi_t^p + (1 - \alpha) \pi_t^c,
\]

where small letters are percentage deviations of relevant variable from their steady state, i.e. \( \pi_t^w = \frac{P_t^w - \bar{P}_t^w}{\bar{P}_t^w} \), the parameters are convolutions of the parameters of the model and \( z_t^w \) and \( z_t^p \) are exogenous latent variables whose interpretation is made precise below.

The demand block is reduced to a single equation that links the unemployment rate gap to the terms of trade while all the other terms are lumped into the latent variable \( z_t^u \):

\[
u_t - u_t^n = - \theta \left( \pi_t^* - \pi_t^p \right) + z_t^u.
\]

I also assume that the foreign price inflation is driven by an exogenous process \( z_t^f \):
Finally the exogenous latent state variables are assumed to follow an autoregressive process:

$$Z_t = \Psi Z_{t-1} + E_t$$

where $Z_t = [z^w_t, z^p_t, z^s_t, z^u_t]'$, $\Psi$ is diagonal matrix with all roots in the unit circle and $E_t = [\eta^w_t, \eta^p_t, \eta^s_t, \eta^u_t]'$ are structural shocks for the model. The structural shocks being exogenous non-modeled forces such as policy actions or supply shocks and demand shocks. The shocks studied in the DSGE model above are an example of the type of micro-foundation that is commonly used to interpret them. I am reluctant to attribute to the micro-founded shock a greater role than to be disciplined examples. Nevertheless I characterize the response of the model economy to various one-time unanticipated shocks contained in $E_t$. In order to do so I need to solve the log-linear approximation and cast the solution in the following state space model:

$$s_{t+1} = P(\vartheta)s_t + Q(\vartheta)z_t,$$

$$z_t = N(\vartheta)z_{t-1} + \eta_t,$$

$$y_t = R(\vartheta)s_t + S(\vartheta)z_t,$$

where $s_t$ is a vector containing the endogenous state variables $\{\pi^w_t, \pi^c_t, \pi^p_t\}$, $z_t$ is a vector containing the exogenous state variables driven by the structural shocks $\eta_t$ and $y_t$ is a vector containing the control variable $\{u_t\}$. $P(\vartheta)$ and $Q(\vartheta)$ are time invariant matrices whose elements $\vartheta = \vartheta(\theta)$, depend, typically non linearly, on a vector $\theta$ containing the structural parameters describing preferences, technologies and government policies.
specified in the DGSE. The first panel of Figure 7 shows the impulse response to $\eta^u$ which triggers the supply mechanism: higher unemployment leads to lower wages that decrease marginal costs and therefore lower prices improving competitiveness. The second panel of Figure 7 shows the impulse response to the other three shocks $\{\eta^w, \eta^p, \eta^s\}$ which trigger the demand mechanism: an increase in the wage, an increase in the final price or a decrease in the foreign price decrease competitiveness and increase unemployment. The size and the persistence of the response of the endogenous variables to the structural shocks, the Impulse Response Functions, convey the information on the various relevant elasticities to gauge the strength of the mechanism under study. For example how much (and for how long) the unemployment rate increases to improve competitiveness after a negative shock. For the purpose of assessing the role of the shocks we need to estimate the parameters $\vartheta$ of model and identify the shocks which is the core issue addressed in this paper. From a theoretical point of view the State Space model is identified (globally) up to the sign and a normalization of the structural shocks variance. This is easily shown in the appendix and depends on the special structure that the solution of the log-linearized DSGE together with the assumptions of the data generating process of the exogenous state variables imposes on the state space model. In practice data limitations, such as a small sample, require to perform robustness checks. The strategy I will consistently follow is to generate artificial data from the log-linear model and check how well I can identify structural shocks and dynamics by estimating the same statistical model I use on the observed Portuguese data. I simulate 100 times the log-linear model and keep the generated four shocks and the time series for the unemployment rate and the real exchange rate. The simulated estate space model is

$$x_{t+1} = A(\vartheta)x_t + B(\vartheta)\eta_t,$$
\[
A(\theta) = \begin{bmatrix}
P(\theta) & Q(\theta) \\
0 & N(\theta)
\end{bmatrix}, B(\theta) = \begin{bmatrix}
0 \\
I
\end{bmatrix}
\]

\[x_t = [s_t, z_t]', \eta_t = [\eta^u_t, \eta^p_t, \eta^w_t]', \eta^, \eta' = I\]

I then estimate\(^8\) a smaller state space model with only the unemployment and competitiveness time series, \(x_t = [u_t, \pi^*_t - \pi_t, z_{SS}, z_{DD}]'\) and \(\eta_t = [\eta^S_t, \eta^D_t]\), with an identical structure for \(A\) and \(B\) of suitable dimensions. The shift to a smaller model is motivated by the feasibility of the empirical exercise. Obviously with only two observed time series I can only recover two structural shocks. However if shocks can be clustered in “class of shocks”, where a class is defined as regrouping shocks that have not too dissimilar effects on the endogenous variables, the all exercise is still viable\(^9\). Figure 8 shows how well I recover the shocks in a simulated sample of 200 observations. The left upper panel plots one realization of the \(\eta^u\) together with the recovered \(\eta_{SS}\). The right upper panel plots one realization of the sum of \(\eta^u + \eta^p + \eta^s\) together with the recovered \(\eta_{DD}\). The lower left panel shows the histogram of the correlation between \(\eta^u\) and \(\eta_{SS}\) and the lower right panel shows the histogram of the correlations between \(\eta^u + \eta^p + \eta^s\) and \(\eta_{DD}\). Figure shows the same exercise using a sample of 64 observations. I interpret these results encouraging enough to pursue the estimation on the true data.

\(^8\)Estimation is performed by maximum likelihood using the Kalman Filter.

\(^9\)For a precise discussion on the feasibility of identifying “class of shocks” when there are more shocks than observables in the context of structural VAR see the Appendix of Blanchard Quah (1989).
Data

I briefly highlight some of the difficulties and weaknesses of the exercise I perform. The idea is to enable the reader to have a better idea of what can be inferred from the results. Quarterly data for Portugal only start in 1995 making it challenging to estimate large macro models. With four lags, and with the sample ending in the fourth quarter of 2011, this gives me 64 observations per time series which is arguably a small number by any standard. The simulations above are encouraging. However they are performed using a small model, with two observables and two non observables, that contains 11 parameters. To put things in perspectives, to estimate the log-linear model with the four endogenous variables I would need to estimate 36 parameters, a futile task with 64 observations. The existing data make it also difficult (and not attempted in this exercise) to separate private from public sector for certain aggregates such as the number of employees which is used to compute the compensation per worker. This is a concern when the stated scope of the exercise is to estimate the private economy adjustment mechanism. Figure 9 plots the data used in the empirical exercise. The real exchange rate (REER) used is the effective real exchange rate based on GDP deflators. The unemployment rate is the raw data published by INE. Compensation inflation is yearly growth rate of the ratio of total employees compensation and total employees. GDP inflation is the yearly growth rate of the GDP deflator. The first observation is that both the unemployment rate and the REER exhibit strong persistence. Statistically I cannot reject that both series contain a unit root. I therefore use the change in the unemployment rate, Δu_t, and the rate of change of the real exchange,

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10The most important problem faced when applying unit root tests is their poor size and power properties (i.e., the tendency to over-reject the null when it is in fact true and under-reject the null when it is in fact false. This problem occurs because of the near equivalence of non-stationary and stationary (highly persistent) processes in finite samples.
\[ \varepsilon_t = \pi_t^* - \pi_t, \text{ where } \pi_t^* = p_t^* - p_{t-1}^*, \sum^n \omega_t p_{it}, i = \text{euro} - \text{partners}, \omega = \text{Eurostat weights}, p = \ln(P_{GDP}). \] The two inflation rates appear to be stationary in the sample. To organize thoughts I first present the evidence in pieces to interpret it through the lens of the mechanism described above. Namely a competitiveness - unemployment nexus, an unemployment - wage nexus, a wage - price nexus and finally a price competitiveness nexus.

**The unemployment competitiveness nexus**

The first model I estimate, Model I, is a bivariate model containing the unemployment rate and the real exchange rate:

\[ x_t = [D_{ft}, e_t, z_t^T, e_t^P]' \]

\[ \eta_t = [\eta_t^{DD}, \eta_t^{SS}]', \eta_t \eta_t' = I \]

As anticipated above, the reason I focus on such minimal model is that it already requires to estimate 11 parameters (remember we have 64 observations). To give a sense of the information recovered by the empirical exercise I present the impulse response functions of the endogenous variables to both shocks. The structure imposed identifies a shock that commands a positive comovement between unemployment and competitiveness and a second shock that commands a more transitory negative comovement between the unemployment rate and the real exchange rate. Figures 10 and 11 plot the Impulse Response Functions (the confidence intervals are missing) together with the two identified shocks. To convey further intuition on the scope of the exercise, I also generate artificial observations from the estimated model by filtering out one shock at the time. More precisely I construct both the reduced form positive and negative nexus between unemployment and competitiveness by simulating two time series using the
empirical model and feeding it with one identified structural shock at the time\textsuperscript{11}. The result of the exercise is to trace out a reduced form demand (DD) and supply (SS) which are shown in the Figures 12 and 13. The slope of the SS that I obtain by regressing\textsuperscript{12} the competitiveness on the unemployment time series plotted in Figure is equal to 1.2 and is the reduced form elasticity of the internal devaluation: a 1 percentage point increase in the unemployment rate is associated with an improvement in the competitiveness differential of 1.2 percentage points. The slope of the DD gives the reduced form elasticity\textsuperscript{13} of the demand channel: an improvement in competitiveness differential of 1 percentage point is associated with a decrease in the unemployment rate of 0.44 percentage points. The IRFS contains information of the speed of the movement along the DD and the SS, in other words the pace at which the economy adjusts after a shock. Shocks that trace out the DD last for about 10 quarters while shocks that trace out the SS are much more persistent and last for more than 40 quarters. This last observation implies that the estimated internal devaluation mechanism is slow. Before turning to the question how much and for how long can we expect unemployment to increase to restore the external competitiveness, I turn to the labor market and goods market pieces of the mechanism.

The wage unemployment nexus

The second model I estimate, Model II, contains the unemployment rate and my measure of wage (compensation) inflation. The state space model is identical to model I but now \( x_t = [\Delta u_t, \pi_t^w, \varepsilon_t^{DD}, \varepsilon_t^{SS}]' \). The analysis is identical to one presented for the unemploy-

\textsuperscript{11}Formally I obtain \( x_{t+1}^{SS} = \hat{A} x_t^{SS} + \hat{B} [0, \hat{\eta}_t^{SS}]' \), and \( x_{t+1}^{DD} = \hat{A} x_t^{DD} + \hat{B} [\hat{\eta}_t^{DD}, 0]' \), where a hat denotes the maximum likelihood estimate and a tilde the smoothed shocks from the Kalman Filter. The scatter plots in Figure 12 and 13 are obtained using respectively \( x_t^{DD} \) and \( x_t^{SS} \).

\textsuperscript{12}The OLS regression is \( \varepsilon_t^{SS} = 0.064 + 1.24 \Delta u_t^{SS} \), t-stat=26.4, \( R^2 = 0.91 \).

\textsuperscript{13}The OLS regression is \( \varepsilon_t^{DD} = 0.13 - 0.44 \Delta u_t^{SS} \), t-stat=-13.89, \( R^2 = 0.757 \).
ment competitiveness nexus and is summarized by Figures 14-17 that plot the Impulse Response Functions (the confidence intervals are missing) together with the two identified shocks and the reduced form demand and supply. The slope of the DD that I obtain by regressing\(^{14}\) the compensation inflation on the unemployment time series plotted in Figure is equal to 0.37: an increase in wage inflation of 0.37 percentage point is associated with a 1 percentage point increase in the unemployment rate. Notice from the figure that the DD does not appear stable. The scatter plot shows that the pre-euro and post-euro DD are estimated to be different\(^{15}\). In the exercise that I perform this comes from the identified shocks (the pre euro and post euro shocks appear different) but could equally indicate an instability of the elasticities. The post euro sample DD slope estimate is 0.77. The slope of the SS gives the reduced form elasticity\(^{16}\): an increase in the unemployment rate of 1 percentage point is associated with a decrease in wage inflation of 2.33 percentage points. The SS corresponds conceptually to a Phillips curve. The IRFS again contain information of the speed of the movement along the DD and the SS. Shocks that trace out the DD last for about 10 quarters while shocks that trace out the SS are much more persistent and last again for more than 40 quarters.

**The price wage nexus**

The last third model I estimate, Model III, contains both the wage and price inflations and corresponds conceptually to a price setting. The state space model is identical to model I but now \(x_t = [\pi_t, \pi_t^w, z_t^T, z_t^P]')\). The results are again summarized by Figures 18-21 that plot the Impulse Response Functions (the confidence intervals are missing) together

\(^{14}\)The OLS regression is \(\pi_{it}^{pp} = 0.7 + 0.37\Delta u_t^{pp},\) t-stat=3.05, \(R^2 = 0.13\).

\(^{15}\)The OLS regression for the sample 1996.1-1999.4 is \(\pi_{it}^{pp} = 2.24 + 1.27\Delta u_t^{pp},\) t-stat=5.77, \(R^2 = 0.70\) and for the sample 2001.1-2011.3 is \(\pi_{it}^{pp} = 0.35 + 0.7\Delta u_t^{pp},\) t-stat=14.58, \(R^2 = 0.83\).

\(^{16}\)The OLS regression is \(\pi_{it}^{pp} = -0.12 - 2.33\Delta u_t^{pp},\) t-stat=-13.63, \(R^2 = 0.75\).
with the two identified shocks and the reduced form demand and supply. The slope of the DD that I obtain by regressing the compensation inflation on the GDP inflation series plotted in Figure is equal to 1.65: an increase in wage inflation of 1.65 percentage point is associated with a 1 percentage point increase in the GDP inflation. The SS corresponds conceptually to a Price Setting curve. The slope of the SS gives the reduced form elasticity: an increase in the rate GDP inflation of 1 percentage point is associated with a decrease in wage inflation of 4.3 percentage points. Shocks that trace out the DD last for about 20 quarters while shocks that trace out the SS are much more persistent and last again for more than 40 quarters.

**The price competitiveness nexus**

The final piece of the empirical model is the link between price inflation and the real exchange rate, $\varepsilon_t = \pi^*_t - \pi_t$. Given the foreign inflation is taken as exogenous the relation is not estimated.

**Putting the pieces together**

To pay tribute to an old tradition, I consolidate the estimated pieces in one figure with four quadrants: Figure 22. Quadrant I is the competitiveness - unemployment nexus, Quadrant II is the unemployment - wage nexus, Quadrant III is the wage - price nexus and finally Quadrant IV is the price - competitiveness nexus. Consider the north-east panel of Figure (quadrant I) that shows the estimated relations between competitiveness and unemployment. The upward sloping line (in red) shows the estimated SS nexus and the downward sloping line (in green) shows the estimated DD nexus. The panel

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17 The OLS regression is $\pi_{PP}^{*} = 0.26 + 1.65\pi_{PP}^{p}, \text{t-stat}=18.37, R^2 = 0.84.$

18 The OLS regression is $\pi_{PP}^{*} = -0.157 - 4.3\pi_{PP}^{p}, \text{t-stat}=-27.85, R^2 = 0.92.$
also shows the transformed data implicitly used for the estimation of the SS (blue dots) and the DD (yellow dots) which provide some visual intuition on the uncertainty of the estimates. This is the case for the instability of the labor market piece, shown in quadrant II. Notice that in the second quadrant, to be consistent with the figure, compensation inflation decreases going north and decreases going south. Therefore the Phillips curve is the red line (blue dots). The reader can immediately observe that in 2011, the last year of the observations, the economy was navigating in uncharted waters according to the statistical model. This observation suggests that the results must be interpreted with caution. Results show that competitive disinflation might well be at work, although the labor market appears to be a weak piece in the transmission. The nexus wage-prices appear to be precisely estimated and the transmission from labor compensation to final prices occurs surprisingly well. Having all the pieces we can turn to the projection into the future.

**Projecting the path of unemployment, competitiveness and inflation**

The main assumptions behind the projection are that the structure of the economy does not change and no other shocks (policy, external, etc...) occurs. The starting point of the simulation is the state of the economy in the fourth quarter of 2011. More precisely I simulate the estimated models using the last value of the smoothed exogenous state variables as initial conditions. For example the projected value for the change in the unemployment rate and in the real exchange rates are obtained in model I by simulating

\[ x_{t+1}^{\text{sim}} = A(\hat{\theta})x_t^{\text{sim}} \]

using as initial conditions \( x_0^{\text{sim}} = [0, 0, \tilde{z}^{DD}(T), \tilde{z}^{SS}(T)] \), where \( T \) is the last observation in the sample. Figure present the simulations using the three models. The first column, corresponding to Model I, shows the projected path for the change in
the unemployment rate (expressed in quarterly rates) and for the REER (also expressed in quarterly rates). It is of some interest that the projected path for the change in unemployment using Model II, although larger, does not differ substantially from the path implied by Model I. The same applies for the projected path of compensation inflation using Model II and Model III. Table 1 shows the cumulated total projected adjustment expected for the next 20 quarters (5 years) and the next 40 quarters (10 years).

<table>
<thead>
<tr>
<th>variable</th>
<th>20 quarters</th>
<th>40 quarters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>model I: +4.41 model II: +5.58</td>
<td>model I: +6.07 model II: +7.86</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>model I: +4.85</td>
<td>model I: +6.6</td>
</tr>
<tr>
<td>$\pi_w$</td>
<td>model II: -12 model III: -14.3</td>
<td>model II: -16 model III: -22</td>
</tr>
<tr>
<td>$\pi_p$</td>
<td>model III: -8.25</td>
<td>model III: -13</td>
</tr>
</tbody>
</table>

Table 1: Cumulated adjustment after 20 and 40 quarters. Numbers are percentage points.

Model I predicts that in front of the current imbalances in the unemployment-competitiveness nexus, the adjustment will require an increase in unemployment of about 4 percentage points in the first 20 quarters. The adjustment is slow can bee seen in the figure (it converges to equilibrium in about 90 periods), after 40 quarters the unemployment rate increase if of approximately 6 percentage points. Model II predicts a similar but stronger adjustment for the unemployment rate, 5.6 percentage points after 20 quarters and 7.86 percentage points after 40 quarters. After 40 quarters, the projected competitiveness adjustment is of 6.62 percentage points and the projected compensation disinflation is of 16 percentage points. Finally Figure 25 show the projected path for the unemployment rate in 2012 with the actual series. Although the adjustment implied by Model I and Model II are large, the actual unemployment rate increased above the projections. This could indicate that unfavorable shocks and policies have continued to hit the economy or that the pace of adjustment during this crisis is faster than the one estimated, indicating
a possible instability of the parameters or a combination of the two.

**Conclusion**

As anticipated caution is required in interpreting the results. The challenges posed by the data availability and the methodological issues are important. As explained above, I have run tests and simulations to address the small sample issue and the results are encouraging. The estimates show that dynamics coherent with the postulated mechanism are present in the data. Ultimately an increase in unemployment improves competitiveness which in turn will decrease unemployment. Admittedly the latter force appears to be weaker than the former implying that the improvement in competitiveness will only slowly translate into an improvement in unemployment. Overall the required adjustment is large and persistent, reflecting the relatively low adjustment speed of the mechanism and the large initial imbalances. Obviously shocks will happen and policies will be taken and the actual path will be different. Nevertheless the forecasts can be interpreted as a counterfactual path in the absence of new shocks and policies and therefore should be of some guidance in designing policies aimed at rebalancing the external adjustment.

**References**


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Automatic Control, 19, 640–646.


(November 2011), 1995-2032.

39, 577–591.

Figure 1: Portugal in the Euro
Figure 2: The data used in the estimation
Figure 3: Unanticipated productivity shock
Figure 4: Unanticipated labor supply shock (negative)
Figure 5: Unanticipated interest rate shock (negative)
Figure 6: Unanticipated foreign consumption shock
Figure 7: The four shocks in the log-linear model. Using a standard parameter configuration, the supply shock, $\eta^u$, commands a positive comovement between $u$ and $\varepsilon$. The three demand shocks, $\eta^w, \eta^p, \eta^l$, command a negative comovement between $u$ and $\varepsilon$. 
Figure 8: Empirical performance of the empirical model using simulated data. The log-linear model with the four exogenous state variables is simulated 100 times using a sample of 200 observations. I estimate a state space model with only two observables, the unemployment rate and the terms of trade, and two exogenous state variables. The upper panels show one realization: the identified shocks, resulting from the estimation, together with the simulated shocks. The lower panels show the histograms of the sample correlations between the identified and simulated shocks: $\rho(\eta^u, \eta^{SS})$ and $\rho(\eta^w + \eta^t + \eta^p, \eta^{DD})$. 
Figure 9: Empirical performance of the empirical model using simulated data. The log-linear model with the four exogenous state variables is simulated 100 times using a sample of 64 observations which correspond to the observed sample. I estimate a state space model with only two observables, the unemployment rate and the terms of trade, and two exogenous state variables. The upper panels show one realization: the identified shocks, resulting from the estimation, together with the simulated shocks. The lower panels show the histograms of the sample correlations between the identified and simulated shocks: $\rho(\eta^u, \eta^{SS})$ and $\rho(\eta^w, \eta^p, \eta^{DD})$. 

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The real exchange rate used is the effective real exchange rate based on GDP deflators, $e_t = \pi_t^* - \pi_t$, where $\pi_t^* = p_t^* - p_{t-1}^*$, $p_t^* = \sum_i^n \omega_i p_t^{*,i}$, $i = \text{euro partners}$, $\omega = \text{Eurostat weights}$, $p = \ln(p^{GDP})$. In the empirical analysis $\Delta u_t$ is the change of the unemployment rate from the same quarter of the previous year.
Figure 11: IRFS of model I to the two shocks.
Figure 12: Shocks
Figure 14: SS
Figure 15: IRFS
Figure 16: Shocks
Figure 17: DD
Figure 18: SS
Figure 19: IRFS
Figure 20: Shocks
Figure 23: The pieces of the mechanism
Figure 24: The simulated path from the three models.
Figure 25: The simulated path for unemployment
Appendix

Equilibrium Conditions

I list here all the equilibrium conditions of the non linear micro-founded model.

\[
\frac{NFA_t}{P_t} = - \left( \frac{Q_t}{S_t} Y_{H,t} - C_t \right) + \frac{\Pi_{t+1} NFA_{t+1}}{1 + i_t P_{t+1}}
\]

\[
Y_{H,t} = \left( \frac{Q_t}{S_t} \right)^{-\eta} \left[ (1 - \alpha) C_t + \alpha \int_0^1 (Q_i)^\eta (S_{t,i} S_{t,i}^i)^{\gamma - \eta} C_i d_i \right]
\]

\[
Q_t = \left[ (1 - \alpha) S_t^{\eta - 1} + \alpha \right]^{\frac{1}{\eta - 1}}
\]

\[
\Xi_{t+1} = \left( \frac{1 + i_t}{1 + i_t^s} \right)^{\frac{1}{2}} \Xi_t
\]

\[
C_t = \Xi_t \left( \frac{1 + \tau_{c,t}}{1 + \tau_{c,t}^s} \right)^{\frac{1}{s}} Q_i \frac{1}{s} C_t^s
\]

\[
N_t = \frac{Y_{H,t}}{A_{H,t}} \Delta_t
\]

\[
\Delta_t = (1 - \delta_p) \left[ \frac{1 - \delta_p \Pi_{H,t}^{e-1}}{1 - \delta_p} \right]^{\frac{1}{e}} + \delta_p \Pi_t^e \Delta_t - 1
\]

\[
(C_t)^{-\sigma} = \beta (1 + i_t) \frac{1 + \tau_{c,t}}{1 + \tau_{c,t+1} P_{t+1}} \frac{P_t}{P_{t+1}} (C_{t+1})^{-\sigma}
\]
\[ 1 + \nu_t = 1 + \nu_t^* + \psi \left( e^{NFA_i/P_t - 1} \right) \]

\[ F_{p,t} = Q_t S_t^{-1} (1 + \tau_{c,t})^{-1} C_t^{-\sigma} Y_{H,t} + \delta_p \Pi_{H,t+1}^e \beta F_{p,t+1} \]

\[ K_{p,t} = \mu_p (1 + \tau_{w,t}) \frac{w_t}{P_t A_{H,t}} (1 + \tau_{c,t})^{-1} C_t^{-\sigma} Y_{H,t} + \delta_p \Pi_{H,t+1}^e \beta K_{t+1} \]

\[ 1 = \delta_p \Pi_{H,t-1}^e + (1 - \delta_p) \left( \frac{K_t}{F_t} \right)^{1-\psi} \]

\[ \Pi_t = \frac{P_t}{P_{t-1}} = \Pi_{H,t} S_t \frac{Q_{t-1}}{Q_t} S_{t-1} \]

\[ K_{w,t} = \mu_w \chi N_t^{1+\phi} + (\delta_w \beta) (\Pi_{w,t+1})^{e_w} (1+\phi) K_{w,t+1} \]

\[ F_{w,t} = (1 + \tau_c)^{-1} C_t^{-\sigma} N_t \frac{W_t}{P_t} + \delta_w \beta (\Pi_{w,t+1})^{e_w^{-1}} F_{w,t+1} \]

\[ 1 = \delta_w \Pi_{w,t-1}^{e_w^{-1}} + (1 - \delta_w) \left( \frac{K_{w,t}}{F_{w,t}} \right)^{1-\frac{e_w}{1+\phi e_w}} \]

\[ L_t = \left( (1 + \tau_{c,t})^{-1} \chi^{-1} \frac{W_t}{P_t} C_t^{-\sigma} \right)^{\frac{1}{\phi}} \Delta_{w,t} \]

\[ \Delta_{w,t} = \delta_w (\Pi_{w,t})^{-\frac{1}{\phi}} \Delta_{w,t-1} + (1 - \delta_w) \left( \frac{K_{w,t}}{F_{w,t}} \right)^{\frac{1}{\phi(1+\phi e_w)}} \]

\[ U_t = L_t - N_t \]
The log-linear approximations, where small letter is the log deviation of a variable from the steady state and capital letters without time subscript indicate a steady state value, are (in the same order):

\[
\begin{align*}
nfa_t &= -Q_s \frac{Y_H}{S_{\text{NFA}}} (q_t - s_t + y_{h,t}) + C_{\text{NFA}} c_t + \frac{\Pi}{1+i} (\pi_{t+1} - r_t + nfa_{t+1}) \\
y_{h,t} &= -\eta \left( \frac{Q}{S} \right) (q_t - s_t) + \frac{(1 - \alpha) C}{(1 - \alpha) C + \alpha Q^n C^*} c_t + \frac{\alpha Q^n C^*}{(1 - \alpha) C + \alpha Q^n C^*} c_t^* + \frac{\alpha \eta Q^n C^*}{(1 - \alpha) C + \alpha Q^n C^*} q_t \\
q_t &= (1 - \alpha) \left( \frac{S}{Q} \right)^{\eta-1} s_t \\
\xi_{t+1} &= \frac{1}{\sigma} \left( \frac{1+i}{1+i^*} \right)^{\frac{1}{\sigma}} (r_t - r_t^*) + \xi_t \\
c_t &= \xi_t + \frac{1}{\sigma} \left( \frac{1 + \tau_c^*}{1 + \tau_c} \right)^{\frac{1}{\sigma}} (\tau_{c,t}^* - \tau_{c,t}) + \frac{1}{\sigma} q_t + c_t^* \\
n_t &= y_{h,t} - a_{h,t} + \delta_{p,t} \\
\delta_{p,t} &= \frac{(1 - \delta_p) e}{\Delta_p} \left( \frac{X}{x} \right)^e (f_t - k_t) + \delta_p \left( \frac{\Pi_H}{X} \right) e (\pi_{H,t} - x_t) + \delta_{p,t-1} \\

\delta_p &= \frac{(1 - \delta_p) e}{1 - \delta_p} \left( \frac{\Pi_H}{X} \right)^{e-1} (\pi_{H,t} - x_t) = (1 - e) (k_t - f_t) \\
c_t &= c_{t+1} - \sigma (r_t - E_t [\pi_{t+1} + \Delta \tau_{c,t+1}] - \rho)
\end{align*}
\]
\[ r_t = \frac{r^*}{r} r^*_t + \frac{\psi NFAe^{NFA}}{r} n f a_t \]

\[ f_{p,t} = \frac{QS^{-1}(1 + \tau_c)^{-1} C^{-\sigma} Y_H}{F_p} (q_t - s_t - \tau_{c,t} - \sigma c_t + y_{h,t}) \]
\[ + \delta_p \left( \frac{\Pi_H}{X} \right)^{\varepsilon - 1} \beta \left( (\varepsilon - 1) (\pi_{H,t+1} - x_{t+1}) + f_{p,t+1} \right) \]

\[ k_{p,t} = \frac{\mu_p (1 + \tau_w) x_H (1 + \tau) C^{-\sigma} Y_H}{K_p} (\tau_{w,t} + \tilde{w}_t - a_{h,t} - \tau_{c,t} - \sigma c_t + y_{h,t}) \]
\[ + \delta_p \left( \frac{\Pi_H}{X} \right)^{\varepsilon} \beta \left( \pi_{h,t+1} - x_{t+1} \right) + k_{p,t+1} \]

\[ \pi_t = \pi_{h,t} + s_t - s_{t-1} + q_{t-1} - q_t \]

\[ k_{w,t} = (1 - \delta_w) (\tilde{\chi}_t + (1 + \phi)n_t) + (\delta_w \beta) \left( \frac{\Pi_w}{X_w} \right)^{\varepsilon_w (1 + \phi)} (\varepsilon_w (1 + \phi) (\pi_{w,t+1} - x_{w,t+1}) + k_{w,t+1}) \]

\[ f_{w,t} = (1 - \delta_w \beta) (n_t + w_t - \tau_{c,t} - \sigma c_t) + \delta_w \beta \left( \frac{\Pi_w}{X_w} \right)^{\varepsilon_w^{-1}} ((\varepsilon_w - 1) (\pi_{w,t+1} - x_{w,t+1}) + f_{w,t+1}) \]

\[ \frac{\delta_w (\varepsilon_w - 1) \left( \frac{\Pi_w}{X_w} \right)^{\varepsilon_w^{-1}} (\pi_{w,t} - x_{w,t})}{1 - \delta_w \left( \frac{\Pi_w}{X_w} \right)^{\varepsilon_w^{-1}}} = \frac{(1 - \varepsilon_w)}{1 + \phi \varepsilon_w} (k_{w,t} - f_{w,t}) \]

\[ l_t = \left[ \frac{1}{\phi} (w_t - \tau_{c,t} - \chi_t - \sigma c_t) + \delta_{w,t} \right] \]

\[ \delta_{w,t} = \frac{\Delta_w}{\Delta_w} \frac{1}{\phi (1 + \phi \varepsilon_w)} \left( \frac{f_{w,t} - k_{w,t}}{1 + \phi \varepsilon_w} \right) + \delta_w \left( \frac{\Pi_w}{X_w} \right)^{-\frac{1}{\varepsilon}} \left( \frac{1}{\phi} (x_{w,t} - \pi_{w,t}) + \delta_{w,t+1} \right) \]

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and where for simplicity \( u_t \) is

\[
u_t = (I_t - n_t).
\]

By combining the log-linear approximations (it is easy to show that \( \bar{\delta}_{u,t} = \bar{\delta}_{p,t} = 0 \)), you obtain the log-linear model presented in the paper.

**Identification of the linear model**

\[
x_{t+1} = A(\vartheta)x_t + B(\vartheta)e_t,
\]

\[
y_t = C(\vartheta)x_t
\]

The identification for this stationary model has been studied in the Engineering literature at least since Kalman and has a definitive treatment in Hannan and Dietmar (1988). Recently Ng (following Glover and Willems (1974) and have adapted it to the identification of DSGE model. By back substitution I can rewrite

\[
y_t = C(\vartheta) \left[ IL + A(\vartheta)L^2 + A(\vartheta)^2L^3 + \ldots + A(\vartheta)^{t-1}L^t \right] B(\vartheta)e_t
\]

and the covariance of the observations:

\[
\text{Cov}_t(y_t) = \Omega_t(\vartheta) = W_t(\vartheta)W_t(\vartheta)'.
\]

The model is identified if \( \Omega_t(\vartheta) = \Omega_t(\psi) \) implies \( \vartheta = \psi \). Assuming that \( \eta_t \eta_t' = I \), the identification condition boils down in showing that for the only square matrix \( T \) and an
orthonormal matrix $U$ that satisfy the following system

$$A(\vartheta)T = TA(\psi), C(\vartheta)T = C(\psi), B(\vartheta)U = TB(\psi)$$

are $T = I$ and $U = I$. In our setting $A(\vartheta) = \begin{bmatrix} P(\vartheta) & Q(\vartheta) \\ 0 & N(\vartheta) \end{bmatrix}$, $B(\vartheta) = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $C(\vartheta) = [H(\vartheta)]$. The first restriction in our system is that we observe the endogenous state variables, we can therefore rewrite $C(\vartheta) = \begin{bmatrix} R(\vartheta) & S(\vartheta) \\ I & 0 \end{bmatrix}$ and partition $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}$ accordingly. It is easy to see that $T_1 = I$ and $T_2 = 0$. The second set of restriction comes from the special structure of $B$. We see that $U = I$ and $T_4 = I$. Finally from we find that $N(\vartheta)T_3 = T_3P(\psi)$ which can be written as $(I \otimes N(\vartheta))vec(T_3) = (P(\psi)' \otimes I)vec(T_3)$. Assuming that $P$ and $N$ do not share any common root implies $T_3 = 0$. 

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