Rhizomatic Thinking and Voting Equilibria in Large Multi-Candidate Elections under Plurality Rule

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Abstract

We consider a model of strategic voting behavior in large multi-candidate elections under Plurality Rule where we allow for the possibility of rhizomatic thinking (Bravo-Furtado & Côrte-Real [4]). Our rhizomatic assumption states that each agent may condition her optimal response on an exogenous belief she possesses over the proportion of like-minded others that will take the same action as she does. In our pivotal-agent game, we therefore relax self-goal choice - and rhizomatic beliefs will induce agents to perceive that they may be pivotal with a non-vanishing probability. We modify the trinomial pivotal-voter model of Palfrey [26] and solve for asymptotic equilibria using techniques from large deviations theory, given the distributions of preferences and beliefs. We show existence and possible uniqueness of equilibria in this setting. We then conclude that our model may help select equilibria, adding predictive power to standard game-theoretic settings. We also find that Duverger’s Law may be violated in equilibrium and, as an application, we suggest rhizomatic thinking can help provide a compelling rationale for the non-Duvergerian outcome of 1992 US presidential election. Correlations between rhizomatic beliefs and preferences explain the different equilibrium outcomes and our model therefore calls attention to the role of partisanship and group identity in plurality election outcomes.

JEL classification: A12, C72, D03, D72

1 Introduction

Large elections have an obvious importance in democratic life and the design of optimal voting rules must take into account its impact on voting behavior. Standard notions of individual rationality may however fail to fully account for observed behavior by voters in large elections.

Individual rationality seems unable to provide a rationale for both instrumental strategic voting behavior as well as for turnout, in very large pivotal-voter equilibrium games. This is due to the fact that, since the probability that any given agent may be pivotal asymptotically vanishes - in all but knife-edge cases -, so should instrumental motivations when compared to expressive ones.

It is important to stress the meaning of the emphasis on the individual component of the rationality assumption, rather than on the bounded component of potentially individual bounded-rationality: more collective notions of rationality may provide a strong additional rationale for instrumental strategic behavior by voters, since these may no longer perceive their pivot probability as infinitesimally small, which may therefore enhance the plausibility of such instrumental behavior. However, instead of considering this feature as a departure from rationality, some authors argue that the correct rationality concept has to be thought of as an integrating part of the context that it is used in (Sen [32], [35] and Acevedo and Krueger [1]).

In a recent paper, Charness, Rigotti and Rustichini [6], claim that "When society is organized in groups, social psychology, sociology, and anthropology hold the view that human behavior is better understood as social behavior. Standard economic analysis has so far typically assumed the opposite. Yet if group membership affects behavior greatly, then we should call into question theories based on the presumption that behavior

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1We will not address turnout in this essay: for a survey, see Dhillon & Peralta [10]. Thorough arguments for somewhat loose claims made in this section can be found in the sequel.
The perception of simultaneity involved in the act of voting may also imply a perception of connection among actions, and a feeling of group identity may indeed play a significant role in this process, even though self-interest may well continue to drive agents’ behavior. The tension between individual and collective notions of rationality becomes particularly prominent in such collective action settings.

Our main purpose is to research the impact that the concept of Rhizomatic Thinking (RT) introduced by Bravo-Furtado and Côrte-Real [4] can have in this setting.

In Biology, "rhizome" is the term for a usually underground horizontal stem of a plant that often sends out roots and shoots from its nodes. Deleuze and Guatari [8] generalize the concept of a rhizome as an acentered and non-hierarchical system made up of possibly heterogeneous connections. We borrowed this notion of a rhizome to introduce the concept of RT. Our rhizomatic behavior assumption states that each agent $i$ may, to various degrees, condition her optimal response on an exogenous rhizomatic belief (RB), $q_i \in [0,1]$, she possesses over the proportion $q_i$ of "like-minded" others that will take the same action as she does. Each agent therefore interprets her own decision as a diagnosis of the decisions that this proportion of "like-minded" others will take. The class of "like-minded" others is game specific, and in most cases it corresponds to the class of agents who share that agent’s preferences. If each agent’s RB is taken to be zero, we recover the standard game-theoretic framework and our assumption can therefore be seen as a generalization of the classical assumption.

We have argued that this may be a unifying assumption on rationality that allows for a general application in games (Bravo-Furtado and Côrte-Real [4]), and that may be particularly useful in collective action settings. If this is the case, RT may provide a strong additional rationale for instrumental behavior by agents in pivotal-agent games (PAG), especially when the number of players is very large. In such cases, game theory prescribes that purely individually rational agents condition their actions on the event that they are pivotal (an event with an infinitesimal likelihood). It is questionable to what an extent such a rationale may plausibly explain relevant instrumental motivations for voluntary participation and/or strategic behavior in large elections when compared to expressive motivations, for example. However, rhizomatic agents may no longer perceive their pivot probability as infinitesimally small, which may therefore enhance the plausibility of such instrumental behavior. RT may therefore add explanatory and predictive power and produce results that differ substantially from those of standard game-theoretic models. Our purpose is to study the effects of introducing RT in PAG, namely by modeling voting behavior in large multi-candidate elections under Plurality Rule (PR).

We consider the population of voters who actually do vote or, equivalently, we do not allow for abstention. It is important to note that RB not only provide agents with additional instrumental motivation to vote strategically but also to participate in elections. However, whereas strategic decisions regarding candidate choice cannot be justified by expressive motivations, it is more difficult to isolate the effects of expressive motivations and rhizomatic thinking with respect to participation. Although it is debatable whether the two decisions are separable, we therefore choose to focus our analysis on candidate choice.

Our model is based on the trinomial model presented by Palfrey [26]. The key difference is that we now endow each agent with a RB besides the standard linear ordering over candidates, where both preferences and the distributions of RB are common-knowledge. The multinomial model turns out to be most appropriate in order to incorporate RT as well as other alternative behavioral assumptions. Our model can be seen as an extension of the classical assumption.

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2 How RB may be formed is of course a pertinent, although largely open and debatable, issue: as is the formation of preferences. On one hand, we would argue that RB, as well as preferences, have an intrinsic exogenous component that is part of an agent’s *privateness* (Sen [35]). RB may be formed in much the same way as other idiosyncratic features of an individual’s privateness which, in particular, result partially from the same environmental variables (education, culture, information, socioeconomic status, etc.). But idiosyncratic features of an agent’s privateness are usually also correlated with each other, and it is also debatable whether they result solely from one’s own environment or not - one’s values and ideologies are correlated with one’s preferences, for example. Moreover, RB are only meaningful on a context dependent basis, since these are liable to vary substantially across different contexts. It seems therefore to be difficult to provide a rationale that predicts that a given RB should result as a function of other features of one’s own privateness. However, it may in general be possible to extract distributions of rhizomatic beliefs compatible with data: Bravo-Furtado and Côrte-Real [4] did so, in the context of the Traveler’s Dilemma.

3 In PAG, we should expect RT to repercute in the models in very much the same way that any behavioral assumption that...
extension of Palfrey’s, which results from setting all RB equal to zero. However, our main contribution may be 
the particular nature of this extension: we enrich the model by adding diversity from a behavioral viewpoint, 
rather than by adding informational uncertainty. The latter approach has widely been undertaken in the 
literature (see Myatt [22] for a recent example). It is also worth remarking that solving our model analytically 
for asymptotic equilibria requires different techniques from those used by Palfrey [26]. Fortunately, large 
deviations theory provides fully adequate and easier-to-handle tools to this effect, which are required in order 
to obtain accurate approximations for the tails of the multinomial distribution.

Palfrey’s main result establishes Duverger’s Law as a regularity: only two-party equilibria arise as possible 
outcomes, since voters who prefer the least voted-for candidate respond by voting for their second-ranked 
candidate in any equilibrium, whereas all other voters vote for their top-ranked candidate. In fact, as 
Laslier [17] points out, any classical pivotal-voter model in which the distribution of preferences is common-
knowledge – such as Myerson’s model of Poisson population uncertainty ( [23], [24] and [25]) -, will produce 
the same result under PR. In our model, however, the rhizomatic best responses of voters who prefer the 
least voted-for candidate will generically depend on their RB in any equilibrium (whereas all other voters 
still vote for their top-ranked candidate). Such best responses are characterized by cutoff levels of the RB for 
each type, which are a function of the distributions of preferences and RB, depending also on the equilibrium 
probabilities. Voters who prefer the least voted-for candidate and whose RB is above the threshold level for 
her type will now vote for their most preferred candidate. This is the first main result of the paper and 
allows us to characterize the equilibria of the voting game.

We then show existence of equilibria and proceed to analyze examples that allow for predictions that differ 
from Palfrey’s. On one hand, uniqueness of equilibrium is possible and unique orderings can be achieved even 
if there is no unique equilibrium: equilibrium selection will in general result from RT, refining the standard 
pilethora of equilibria in large elections and endowing the theory with a stronger predictive power.

We also conclude that if there is one candidate who is preferred by at least half of the voters and more 
than \( \frac{2}{3} \) of these voters share the same preference ordering, then that candidate always wins provided agents 
are rhizomatic "enough"; in this particular case, we eliminate counter-intuitive Duvergerian equilibria in 
which such an alternative would come in last. Moreover, three-party equilibria may occur and Duverger’s 
Law may fail to hold. This is an important consequence of our model and it may help explain observed non-
Duvergerian outcomes in plurality elections. This may help explain why Duverger’s Law is not a regularity 
in elections in India (where in fact it hardly ever holds) and in constituency-level elections in the UK, to 
cite the most well-known exceptions (Myatt [22]). It can also provide a rationale for other examples such as 
the 1992 US presidential election where Ross Perot received a significant proportion of the popular vote in 
almost all states. We discuss in detail the three-candidate outcome of the Bush-Clinton-Perot presidential 
race and show that RT and correlations between preferences and beliefs can add insight to this particular 
case.

In general, vote manipulation is (weakly) decreasing in RB but the driving forces that may bring about 
different equilibrium outcomes are the distributions of preferences and rhizomatic beliefs, including the 
respective correlations. We therefore suggest that besides preference formation, there is another potential 
field of interest for political or party strategists: campaigns may influence the formation of rhizomatic 
beliefs, whether by creating feelings of group identity among partisans or by appealing to strategic voting 
from supporters for other parties.

The paper proceeds as follows.

Section 2 reviews the literature. Due to the multi-disciplinary scope of our analysis, we split it twofold. 
We first examine the determinants of voting behavior and argue that RT can capture alternative rationales 
proposed in the literature to explain strategic voting behavior in PAG. We then proceed to focus on the 
regularity of Duverger’s Law. We also review Bravo-Furtado and Côrte-Real [4] and argue in detail why 
we find RT to be an appropriate and unifying notion of rationality for modeling behavior in games and 
specifically in collective action settings such as PAG.

In section 3 we present and solve our model asymptotically. We specify the best responses for all agents 
and characterize the asymptotic equilibria resorting to technical results from large deviations theory (results 
that we summarize in appendix A). We proceed to discuss the main characteristics of our results. We then

\(^4\)If any exist, and whenever this threshold is below 1.
argue that simple discrete distributions of RB can be used to capture fairly general distributions with no loss of generality and with considerable gains in terms of insight. This allows us to understand equilibrium selection and the effects of correlations between RB and preferences.

Section 4 studies illustrative examples that allow us to understand the mechanisms that drive equilibrium selection and to achieve possible policy implications.

Section 5 concludes and summarizes the main findings.

2 Literature Review

Standard game theory assumes - and prescribes - individual rationality as the sole determinant of each agent’s behavior. According to Sen [35], individual rationality is characterized by three independent and complementary assumptions: self-centered welfare ("a person’s welfare depends only on his or her own consumption"), self-welfare goal ("a person’s only goal is to maximize his or her own welfare" and not "the welfare of others"), and self-goal choice ("each act of choice of a person is guided immediately by the pursuit of one’s own goal" and "in particular, is not restrained by the recognition of other people’s pursuit of their goals"). Several authors have long criticized this assumption, namely in collective action settings, on both positive and normative counts (see Sen [32], [34], [35]). However, no alternative unifying concept of rationality has so far been formalized in order to address this criticism.

A related line of criticism with respect to the concept of individual rationality, pointed out by Sen [33] and Gilboa et al. [14], is its applicability to large games such as elections. In the past two decades, studies of large elections under game-theoretic settings (such as Palfrey’s [26]) have provided strong theoretical support in favor of Duverger’s Law: in a three-candidate election, there would only be two-party equilibria. However, empirically this is not the only observed outcome – constituency-level elections in the UK and India are well-known exceptions. One theoretical exception to the prediction of Duvergerian outcomes is that of Myatt [22]. Myatt’s approach is based on signals and informative voting and his three-party equilibrium predictions rely solely on voters’ uncertainty with respect to the identity of the leading defiant candidate. Myatt’s contextualization therefore requires that polls, or alternative informational mechanisms, are sufficiently uninformative about the relevant priors to all voters - and other settings would still lead to Duvergerian outcomes.

Standard notions of individual rationality applied to large elections seem to be unable to explain non-Duvergerian outcomes but also fail to explain why only certain focal equilibria emerge in practice - and are unable to explain turnout as well. Recent voting literature has started to question "full" individual rationality (see Dhillon and Peralta [10] for a survey of the literature on turnout), as well as the connection between preference and choice (see, for instance, Côrte-Real [7]). Over the past few years, political scientists (Felsenthal et al. [13], Felsenthal [12], McKelvey and Ordeshook [18] and [20], Rietz [28], Scotto and LaFone [30]) have also shown a growing interest in the effects of strategic voting behavior on coordination and political representation in multi-candidate elections, for which there is abundant empirical and experimental evidence - but that still lacks an adequate theoretical model.

Other scientific branches suggest alternative rationales for collective action. Agents may believe that their voting decision induces others to vote likewise (magical thinking). Alternatively, agents may merely believe that their own actions are a diagnosis of collective behavior, that is, they may read their own vote as a sign that many like-minded others will vote too, and perhaps with the same voting pattern ("illusory correlation" - beliefs that inaccurately suppose a relationship between a certain type of action and an effect).\footnote{It is important to stress the different mental mechanisms behind the "illusory correlation" bias and RT. Illusory correlation inaccurately presupposes a relationship between a certain type of action and an effect. Our rhizomatic assumption is weaker and horizontal: the action is constrained to the perception that a proportion of other agents will act accordingly.}

Goldberg et al. [15] suggest individual agents may follow a "what if everyone acted that way" type of reasoning, referred to in the Cognitive Sciences literature as the "voter’s illusion" or, equivalently, as the mixture of "symmetry" and the "illusion of control".

With the paper by Brams [3] on Newcomb’s problem and the prisoner’s dilemma, and the influential work of Quatrone and Tversky [27], collective reasoning - namely the "voter’s illusion" - has emerged as a likely determinant for strategic voting behavior and coordinated outcomes. It is the one for which experimental evidence is most consistent (Morris et al. [21], Acevedo and Krueger [1]) and provides a rationale for strategic
behavior with less demanding computational requirements. In its weakest version, agents consider their own behavior as a diagnosis of the behavior of like-minded others (agents with the same preferences or in the same "group"). However, instead of considering this feature as a departure from rationality, some authors argue that the correct rationality concept has to be thought of as an integrating part of the context that it is used in (Acevedo and Krueger [1]).

Moreover, Charness et al. [6] conclude, in an experimental setting, that the perception of group salience and group membership influence strategic behavior in Matching-Pennies and prisoner’s dilemmas games.

Also, models that allow for explicit individual bounded-rationality seem to be of little use in explaining agents’ strategic behavior in large collective action games.

One such example is the hybrid model by Camerer, Palfrey and Rogers [5], that captures both cognitive hierarchy (CH) and quantal response (QR) equilibrium models. However, CH models are not computable for large games, and QR models assume that players have decreased payoff responsiveness. The correct direction for any strategic pivotal model of voting capable of increasing predictive power seems to demand an assumption that translates into the model by making agents act as if they were more often pivotal than what they really are.

In Bravo-Furtado and Corte-Real [4], we argued extensively why RT may be an appropriate concept to model collective action, particularly in large games, reconciling individual and more collective notions of rationality.

An agent will think rhizomatically if she believes that a proportion of like-minded others will take the same action as she does. The agent therefore perceives a connection (that is actually "inexistent") among herself and like-minded others. By allowing for RT, we are therefore allowing for the mental mechanism to be such that the agent believes that she belongs to a network with no actual links, and such that she acts in accordance with that belief. We thus add a new variable to the classical game-theoretic setting: an exogenous belief that is an idiosyncratic feature of the agent i.e. a feature of an agent’s "privateness" (Sen [35]).

Anderson [2], in his analysis of nationalism, states that "A nation is "an imagined political community". It is imagined because the members of even the smallest nation will never know most of their fellow-members, meet them, or even hear of them, yet in the minds of each lives the image of their communion. (...) In fact, all communities larger than primordial villages of face-to-face contact (and perhaps even these) are imagined.". The idea of an imagined community is one of the components of RT; the second important component is the restriction on an agent’s action imposed by this "imagined community".

A person who is reading a book while waiting for a medical appointment and who decides to close the book at a given moment may perceive that several other people around the world performed that same action at the same time, and yet perceives no connection among those actions. On the other hand, in a collective action context such as voting, the perception of simultaneity may also involve the perception of connection among actions. This example suggests that mental processes matter and are liable to depend on specific contexts, like Rubinstein and Salant [29] point out. RT seems capable of encompassing this diversity and context-dependence in a general and unifying way.

We should stress that RT relaxes only the self-goal choice assumption, even though self-interest may well continue to drive agents’ behavior – and it encompasses individual rationality as well as a special case. Moreover, RT can also capture the ideas of collective reasoning, mutual interdependence, group identity (and "impure altruism"), as well as cognitive biases such as the belief in personal relevance, the voter’s illusion and the illusion of control.6

The tension between individual and collective notions of rationality becomes particularly prominent in collective action settings. In order to address several open questions regarding voting behavior in a large election (where group identity may indeed play a significant role), namely to examine the validity of Duverger’s Law and the selection of focal equilibria, we therefore introduce the concept of RT to model a large voting game under PR.

6In any case, mental processes are very different under each assumption. The “voter’s illusion” cognitive bias implies that agents actually perceive an “illusion of control”: their decision is perceived as influencing others to act likewise, whereas RT is a horizontal notion.
3 The Model

We first define the fundamental concepts.

Consider a normal form game with \( n \) players, independently drawn from a population of \( N \) agents. Let \( S_i \) denote the space of pure strategies and let \( \Sigma_i = \Delta(S_i) \) be the space of mixed strategies of player \( i \). Let \( N_i \subseteq N \) be the game-specific set of player \( i \)’s like-minded agents. Let \( q_i \in Q_i \subseteq [0,1] \) be the rhizomatic type of agent \( i \), that is, the proportion of agents in \( N_i \) that agent \( i \) believes will take the same action as she does. Let \( \theta_i \) stand for all components of \( i \)’s type other than \( i \)’s rhizomatic belief. Player \( i \)’s type can then be denoted by \( t_i = (\theta_i, q_i) \). We assume that \( t_i \) is known only to player \( i \). Let \( F \) denote a (common-knowledge) cumulative distribution function (cdf) of types and let \( F_i \) denote the marginal cdf for player \( i \).

Definition 1 A rhizomatic strategy for agent \( i \) is a map \( p_i : T_i = \Theta_i \times Q_i \rightarrow \Sigma_i \).

Let \( u_i(p_i, p_{-i}, t_i | t_i) \) denote player \( i \)’s expected payoff associated with playing \( p_i \) when all others are playing \( p_{-i} \), given that her type is \( t_i \). As usual, \( -i \) denotes \( j = 1, ..., n \) with \( j \neq i \).

Note that the choice of terminology is meant only to emphasize the exogeneity of beliefs \( q_i \), which are now part of an agent’s type: in all other regards, the definitions are equivalent to those in Bayesian Nash settings.

Definition 2 A rhizomatic strategy \( p_i \) is a Rhizomatic Best Response (RBR) to \( p_{-i} \) if for all \( p_i' \) and for all \( t_i' \)

\[
E_{t_i}(u_i(p_i(t_i'), p_{-i}(t_{-i}), t'_i) | t'_i) \geq E_{t_i}(u_i(p_i'(t_{-i}), t'_i) | t'_i)).
\]

Definition 3 A Rhizomatic Nash Equilibrium (RNE) is a vector of rhizomatic strategies \( (p_1, ..., p_n) \) such that, \( \forall i, p_i \) is a RBR to \( p_{-i} \).

Definitions 2 and 3 characterize the equivalent of Bayesian Nash Best Response and Bayesian Nash Equilibrium under a common-knowledge distribution of priors with private realizations of types.

We can now present the model.\(^7\)

There are three candidates, \( A, B \) and \( C \), and polity with a large number \( N \) of voters.

Each voter \( i \) has a strict preference ordering over the three candidates. A voter’s preference for each candidate is represented by a Von Neumann-Morgenstern utility number, normalized so that voter \( i \) receives a utility of 1 if her preferred candidate wins, a utility of 0 if her least preferred candidate wins, and a utility of \( \lambda_i \in (0,1) \) if the winner is her middle-ranked candidate.

Each voter also possesses an exogenous rhizomatic belief, \( q_i \in [0,1], \) as to the proportion of like-minded others she believes will always choose the same action as she does.

Let a \( jk \)-voter denote a voter whose preferred candidate is \( j \) and whose second-preferred candidate is \( k \). Let \( l_{jk} \) represent the average proportion of \( jk \)-voters in the electorate. Hence, \( l_{AB} + l_{AC} + l_{BA} + l_{BC} + l_{CA} + l_{CB} = 1 \).

Let a \( jk \)-type denote a \( jk \)-voter whose utility if \( k \) wins is \( \lambda_i \), and whose rhizomatic belief is \( q_i \). Let \( F_{jk}(\ldots) \) represent the cumulative distribution function (cdf) of \( jk \)-types.

We assume that \( l_{jk} \) and \( F_{jk}(\ldots) \) are common knowledge, for all \( j, k \in \{A, B, C\} : j \neq k \), and that each voter’s type is drawn independently from the probability distribution \( P = (l_{AB}, ..., l_{CB}, F_{AB}, ..., F_{CB}) \).

Let \( \tilde{N}_{jk} \) denote the actual realization of the number of \( jk \)-types in the election, which voters do not know. The relevant rhizomatic class of "like-minded" others for a \( jk \)-voter will be the class of other \( \tilde{N}_{jk} - 1 \) \( jk \)-voters plus herself. We will show ahead that this uncertainty is asymptotically resolved, and that a \( jk \)-type’s asymptotic rhizomatic best response will be a function of \( l_{jk} \) and not of \( \tilde{N}_{jk} \).

We also assume that no voters abstain and introduce the following additional assumptions:

Assumption 1 \( l_{jk} > 0 \), for all \( j, k \in \{A, B, C\} : j \neq k \).

\(^7\)For ease of comparison, we follow the notation of Palfrey [26] closely - with minor adjustments that our setting requires. In an abuse of notation, we consider henceforth the players to be \( N \).

6
Rhizomatic Best Response for any voter to her, for an actual realization of types. Let

## 3.1 Plurality Rule (PR)

The voting game is standard. Under PR, each voter simultaneously chooses one candidate to vote for and the candidate with the highest number of votes will win. Following Palfrey [26], and for simplicity, ties will be broken alphabetically (e.g. A beats B in a tiebreaker). This assumption will not affect our results since ties are zero-measure events when \( N \to \infty \).

We will look for (pure strategy) symmetric Rhizomatic Nash Equilibria, i.e. Bayes-Nash equilibria conditioned on a voter’s type (which includes her rhizomatic belief), in which no voter chooses a weakly dominated strategy. A (pure) strategy for agent \( i \) under PR is therefore a measurable function

\[
\sigma_i^{PR} : \{AB, AC, BA, BC, CA, CB\} \times (0,1) \times [0,1] \to \{A, B, C\}
\]

In a pure-strategy symmetric Rhizomatic Nash Equilibrium, \( \sigma_i^{PR} = \sigma_{PR} \) for all \( i \).

### 3.1.1 Assumption 2

Assumption 2 states that all possible preference rankings occur: this is a minor technical assumption and its only purpose is allowing to state our results for all possible polities. It can, however, be easily relaxed in a straightforward manner without affecting the results.

Assumption 2 rules out mass points in \( \lambda_i \) and in \( q_i \in (0,1) \). It ensures it is a zero-measure event for two voters to have the same preference ordering and the same intensity of preference \( \lambda_i \) (this can also be straightforwardly relaxed in our asymptotic setting). However, we allow for mass points in \( q \in (0,1) \). On one hand, we would like to allow for a proportion of voters not to be rhizomatic at all; on the other hand, this allows us to recover the standard framework (if this proportion is equal to 1). The possibility of mass points at \( q = 1 \) will not affect our results and will allow us to analyze examples of discrete distributions as well.

We are also omitting the subscript \( jk \) and in \( \lambda \), \( q_i \) and \( p \).

### Assumption 2

\( F_{jk}(\cdot, \cdot) \) is twice continuously differentiable for \((\lambda_i, q_i) \in (0,1) \times (0,1)\) and \( \frac{\partial^2}{\partial \lambda^2} F_{jk}(\lambda, q) > 0, \forall (\lambda, q) \in (0,1) \times (0,1) \), for all \( j, k \in \{A, B, C\} : j \neq k \). Furthermore, \( F_{jk}(0, q_i) = 0, F_{jk}(1, q_i) = G_{jk}(q_i) \) and \( G_{jk}(0) = q_{jk} \in [0,1] \).

Let \( O_{jk} \) be the number of other \( jk \)-type voters who vote for \( A \), \( \bar{b}_{O} \) who vote for \( B \) and \( \bar{c}_{O} \) who vote for \( C \).

We now simplify the notation and remove the subscript \( O \) from the variables.

Hence,

\[\bar{M}_{jk} = 1 + \left[ q_i(N\bar{M}_{jk} - 1) \right]^{8}\]

Let \( \bar{O}_{jk} = N - \bar{M}_{jk} = N - 1 \left[ q_i(N\bar{M}_{jk} - 1) \right] \) be the set of agents that \( i \) does not perceive as “linked” to her, for an actual realization of types. Let \( \pi_j \) denote the probability a randomly selected voter out of \( \bar{O}_{jk} \) votes for \( j \), such that \( \pi_A + \pi_B + \pi_C = 1, \pi_j \geq 0 \). For any \( \sigma_{PR} \) and \( (\pi_A, \pi_B, \pi_C) \), we can characterize a Rhizomatic Best Response for any voter \( i \). We do so, without loss of generality, for an \( AB \)-type.

Let:

\[\bar{p}_{AB} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]

\[\bar{p}_{AC} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } C, \text{ for an actual realization of types.}\]

\[\bar{p}_{CB} = \text{probability that voting for } A \text{ yields } C, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]

The probabilities \( \bar{p}_{AB}, \bar{p}_{AC} \) and \( \bar{p}_{CB} \) result from the trinomial distribution with parameters \( \bar{O}_{jk} \) and \( \pi_j, j = A, B, C \). We derive below these probabilities for an \( AB \)-type, without loss of generality, for an actual realization of types. For ease of notation, we will henceforth omit the subscript \( i \) and also the superscript \( N \) and the subscript \( AB \) from \( \bar{O}_{jk} \) and \( \bar{M}_{jk} \), and also all remaining variables, whenever there is no room for misinterpretation.

Let \( \bar{a}_{\bar{O}} \) be the number of other \( \bar{O} \) voters who vote for \( A \), \( \bar{b}_{\bar{O}} \) who vote for \( B \) and \( \bar{c}_{\bar{O}} \) who vote for \( C \).

We now simplify the notation and remove the subscript \( \bar{O} \) from the variables.

\[\bar{M}_{jk} = 1 + \left[ q_i(N\bar{M}_{jk} - 1) \right]^{8}\]

\[\bar{O}_{jk} = N - \bar{M}_{jk} = N - 1 \left[ q_i(N\bar{M}_{jk} - 1) \right] \]

\[\bar{p}_{AB} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]

\[\bar{p}_{AC} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } C, \text{ for an actual realization of types.}\]

\[\bar{p}_{CB} = \text{probability that voting for } A \text{ yields } C, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]

The probabilities \( \bar{p}_{AB}, \bar{p}_{AC} \) and \( \bar{p}_{CB} \) result from the trinomial distribution with parameters \( \bar{O}_{jk} \) and \( \pi_j, j = A, B, C \). We derive below these probabilities for an \( AB \)-type, without loss of generality, for an actual realization of types. For ease of notation, we will henceforth omit the subscript \( i \) and also the superscript \( N \) and the subscript \( AB \) from \( \bar{O}_{jk} \) and \( \bar{M}_{jk} \), and also all remaining variables, whenever there is no room for misinterpretation.

Let \( \bar{a}_{\bar{O}} \) be the number of other \( \bar{O} \) voters who vote for \( A \), \( \bar{b}_{\bar{O}} \) who vote for \( B \) and \( \bar{c}_{\bar{O}} \) who vote for \( C \).

We now simplify the notation and remove the subscript \( \bar{O} \) from the variables.

\[\bar{M}_{jk} = 1 + \left[ q_i(N\bar{M}_{jk} - 1) \right]^{8}\]

\[\bar{O}_{jk} = N - \bar{M}_{jk} = N - 1 \left[ q_i(N\bar{M}_{jk} - 1) \right] \]

\[\bar{p}_{AB} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]

\[\bar{p}_{AC} = \text{probability that voting for } A \text{ yields } A, \text{ but voting for } B \text{ yields } C, \text{ for an actual realization of types.}\]

\[\bar{p}_{CB} = \text{probability that voting for } A \text{ yields } C, \text{ but voting for } B \text{ yields } B, \text{ for an actual realization of types.}\]
\[ \bar{a} + \bar{b} + \bar{c} = \bar{O} \]

Note that \( \bar{M} + \bar{a} = \bar{b} \wedge \bar{M} + \bar{a} \geq \bar{c} \) are the conditions required for \( A \) winning when \( i \) votes for \( A \), and \( \bar{M} + \bar{a} < \bar{b} \wedge \bar{M} + \bar{a} \leq \bar{c} \) are the conditions for a victory of \( C \) when \( i \) votes for \( B \). Thus, \( \bar{p}_{AC}^N \) is the probability

that out of \( \bar{O} \) other voters, \( \bar{a} \) vote for \( A \), \( \bar{b} \) vote for \( B \), \( \bar{c} \) vote for \( C \), and

\( \bar{M} + \bar{a} < \bar{b} \wedge \bar{M} + \bar{a} \geq \bar{c} \wedge \bar{M} + \bar{b} < \bar{c} \wedge \bar{M} + \bar{a} < \bar{c} \wedge \bar{a} \), \( \bar{b}, \bar{c} \geq 0 \).

\( \bar{p}_{AC}^N = \Pr(\bar{c} > \bar{a} \geq \bar{c} - \bar{M} \geq \bar{b} \geq 0) \). Let \( \bar{j} = \frac{1}{2}, \bar{j} \in \{\bar{a}, \bar{b}, \bar{c}\} \). Let

\[ e_{AC}|\bar{O} = \left\{ (\bar{a'}, \bar{b'}, \bar{c'}) : \bar{c'} > \bar{a'} \geq \bar{c'} - \frac{\bar{M}}{\bar{O}} > \bar{b}', \bar{a'} + \bar{b'} + \bar{c'} = 1, \bar{a'}, \bar{b'}, \bar{c'} \geq 0 \right\} \]

Then,

\[
\bar{p}_{AC}^N = \sum_{(\bar{a}', \bar{b}', \bar{c}') \in e_{AC}|\bar{O}} \frac{\bar{O}}{k} \left( \frac{\bar{O} - k}{k + l} \right) \pi_A^k \pi_B^{\bar{O} - 2k - l} \pi_C^{k + l}
\]

We now provide a more explicit expression for \( \bar{p}_{AC}^N \), which will be useful for a graphic comparison with the expression for \( p_{AC}^N \) in Palfray [26].

Let the number of votes for \( A, B \) and \( C \) respectively be \( k, \bar{O} - 2k - l \), and \( k + l \). From the above conditions, we get:

\[ \bar{c} > \bar{a} \iff l > 0 \]
\[ \bar{a} \geq \bar{c} - \bar{M} \iff l \leq \bar{M} \]
\[ \bar{c} - \bar{M} > \bar{b} \iff k > \frac{N - 2l}{3} \]

and finally, \( \bar{b} \geq 0 \iff k \leq \frac{\bar{O} - l}{2} \)

which yields

\[ \bar{p}_{AC}^N = \sum_{l=1}^{\bar{M}} \sum_{k=\left\lfloor \frac{N - 2l}{3} \right\rfloor + 1}^{\bar{O} - k \left( \bar{O} - k \right)} \pi_A^k \pi_B^{\bar{O} - 2k - l} \pi_C^{k + l} \]

In a similar way, \( \bar{p}_{CB}^N \) is the probability that out of \( \bar{O} \) other voters, \( \bar{a} \) vote for \( A \), \( \bar{b} \) vote for \( B \), \( \bar{c} \) vote for \( C \), and

\( \bar{M} + \bar{a} < \bar{c} \wedge \bar{b} < \bar{c} \wedge \bar{M} + \bar{b} > \bar{a} \wedge \bar{M} + \bar{a} \geq \bar{c} \wedge \bar{a} \), \( \bar{b}, \bar{c} \geq 0 \).

\( \bar{p}_{CB}^N = \Pr(\bar{c} > \bar{b} \geq \bar{c} - \bar{M} \geq \bar{a} \geq 0) \). Let

\[ e_{CB}|\bar{O} = \left\{ (\bar{a'}, \bar{b'}, \bar{c'}) : \bar{c'} > \bar{b'} \geq \bar{c'} - \frac{\bar{M}}{\bar{O}} > \bar{a'}, \bar{a'} + \bar{b'} + \bar{c'} = 1, \bar{a'}, \bar{b'}, \bar{c'} \geq 0 \right\} \]

Then,

\[
\bar{p}_{CB}^N = \sum_{(\bar{a}', \bar{b}', \bar{c}') \in e_{CB}|\bar{O}} \frac{\bar{O}}{k} \left( \frac{\bar{O} - k}{k + l} \right) \pi_A^k \pi_B^{\bar{O} - 2k - l} \pi_C^{k + l}
\]

Let the number of votes for \( B \) be \( k \), for \( C \), \( k + l \), and for \( A \), \( \bar{O} - 2k - l \). Then,

\[ \bar{p}_{CB}^N = \sum_{l=1}^{\bar{M}} \sum_{k=\left\lfloor \frac{N - 2l}{3} \right\rfloor + 1}^{\bar{O} - k \left( \bar{O} - k \right)} \pi_A^k \pi_B^{\bar{O} - 2k - l} \pi_C^{k + l} \]

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Also, $P_{AB}^N$ is the probability that out of $\bar{O}$ other voters, $\bar{a}$ vote for $A$, $\bar{b}$ vote for $B$, and $\bar{c}$ vote for $C$, and $\bar{a} \geq \bar{b} \land \bar{M} + \bar{a} \geq \bar{c} \land \bar{M} + \bar{b} \geq \bar{c} \land \bar{M} + \bar{b} \geq \bar{a} \land \bar{M} + \bar{b} \geq \bar{a} \land \bar{M} + \bar{b} \geq 0$.

In this case, it helps splitting the above conditions into two mutually exclusive cases: $\bar{a} \geq \bar{b}$ and $\bar{a} < \bar{b}$. Thus, $P_{AB}^N = \Pr(M + b \geq \bar{a} \land M + b \geq \bar{c} \land \bar{b} \geq 0 \land \bar{c} \geq 0 \land \bar{a} \geq \bar{b}) + \Pr(M + \bar{a} \geq \bar{c} \land M + \bar{a} \geq \bar{b} \land \bar{a} \geq 0 \land \bar{c} \geq 0 \land \bar{a} \geq \bar{b})$. Let

$$e_{AB|\bar{O}} = \begin{cases} (\bar{a}', \bar{b}', \bar{c}') : \bar{M} + \bar{b}' > \bar{a}' \land \bar{M} + \bar{b}' \geq \bar{c}' \land \bar{a}' \geq \bar{b}' \forall \bar{O} \\ \bar{M} + \bar{a}' \geq \bar{c}' \land \bar{M} + \bar{a}' \geq \bar{b}' \land \bar{a}' \geq \bar{b}' \land \bar{a}' \geq \bar{b}' \land 1, \bar{a}', \bar{b}', \bar{c}' \geq 0 \end{cases}$$

Then,

$$\bar{P}_{AB}^N = \sum_{(\bar{a}', \bar{b}', \bar{c}') \in e_{AB|\bar{O}}} \left( \begin{array}{c} \bar{O} \\ k \\ \bar{M} \end{array} \right) \left( \begin{array}{c} \bar{O} - k \\ k + l \\ \bar{M} \end{array} \right) \pi_A^k \pi_B^{\bar{O} - k - l} \pi_C^{k + l}$$

More explicitly,

$$\bar{P}_{AB}^N = \sum_{l=0}^{\bar{M}-1} \sum_{k=\max\{0, R\}} \left( \begin{array}{c} \bar{O} - l \\ k \\ R \end{array} \right) \left( \begin{array}{c} \bar{O} - k \\ k + l \\ R \end{array} \right) \pi_A^k \pi_B^{\bar{O} - k - l} \pi_C^{k + l} + \sum_{l=1}^{\bar{M}} \sum_{k=\max\{0, R\}} \left( \begin{array}{c} \bar{O} - l \\ k \\ R \end{array} \right) \left( \begin{array}{c} \bar{O} - k \\ k + l \\ R \end{array} \right) \pi_A^k \pi_B^{\bar{O} - k - l} \pi_C^{k + l}$$

where $R = \left\{ \begin{array}{ll} \frac{\bar{O} - \bar{M} - l}{3}, & \text{if } \frac{\bar{O} - \bar{M} - l}{3} \notin \mathbb{N} \\ \frac{\bar{O} - \bar{M} - l}{3}, & \text{if } \frac{\bar{O} - \bar{M} - l}{3} \in \mathbb{N} \end{array} \right.$

Note that, as it ought to be the case, the expressions for the probabilities above reduce to those in Palfrey [26] when $q = 0$.

However, a rhizomatic $AB$-type $i$ does not know the realization of $P_{jk|\bar{O}}^N$, $j, k \in \{A, B, C\} : j \neq k$, and, therefore, when computing her expected payoff for each action, she must compute the expected value of each $P_{jk|\bar{O}}^N$ - that is, from her viewpoint, there is population uncertainty as to the actual number of "like-minded" others.

However, we now show that, asymptotically, she best responds as if considering that the number of her "like-minded" others is precisely $M_{AB}^N = 1 + [q_i(N_{AB} - 1)]$ and that $O_{AB}^N = N - M_{AB}^N$ i.e. her asymptotic best response will depend only on the probability $l_{AB}$ of the distribution of preferences in the population.

Let $u_i(J, p_{-i}(t_{-i}), (AB, \lambda_i, q_i)) \vert (AB, \lambda_i, q_i)$ denote the utility of an $AB$-type when she votes for $J \in \{A, B\}$ and $E_{L_{-i}}(u_i(\cdot))$ denote its expected value.

Then, an $AB$-type votes for $A$ if

$$E_{L_{-i}}(u_i(A, p_{-i}(t_{-i}), (AB, \lambda_i, q_i)) \vert (AB, \lambda_i, q_i)) > E_{L_{-i}}(u_i(B, p_{-i}(t_{-i}), (AB, \lambda_i, q_i)) \vert (AB, \lambda_i, q_i))$$

and votes for $B$ if the above inequality is reversed. Now,

\footnote{The lower and upper bounds of the summations in Palfrey [26] contain a few obvious typos.}
(2)
As in Palfrey [26], strategic voting requires λ to be large enough and that the voter is more likely to be pivotal between her second and least preferred candidate than between her first and last.

In order to fully characterize a RNE, we must impose that the values of the probabilities πj, j = A, B, C, are in fact confirmed in equilibrium. An equilibrium is therefore characterized by a set of six cutpoints, \( \lambda^N_{jk}, j, k \in \{A, B, C\} : j \neq k \), each of which must either be 1 or satisfy condition 3 with equality. Since the cutpoints depend on rhizomatic beliefs in a way to be specified ahead, we delay the presentation of the asymptotic equilibrium conditions that confirm the equilibrium probabilities until the necessary results to this effect are established.

### 3.2 Asymptotic Equilibria

Since we are considering very large electorates (\( N \to \infty \)), we can establish the asymptotic properties of the probabilities derived above for a \( jk \)-type with \( q > 0 \) (for \( q = 0 \), the results coincide with Palfrey's) and therefore determine the best responses for each type of voter. The following two propositions are crucial for our results. With no loss of generality, we state them for an \( AB \)-type.

Let \( x = x_{AB} = \frac{q_{AB}}{1 - q_{AB}} \). Let \( \Delta \) denote the two-dimensional simplex. Let:

\[
E_{AC} = \left\{ (\pi_A, \pi_C) \in \Delta : \pi_C \geq \pi_A \wedge \pi_C \leq x + \pi_A \wedge \pi_C \geq \frac{1 + x}{2} - \frac{\pi_A}{2} \right\}
\]

\[
E_{CB} = \left\{ (\pi_A, \pi_C) \in \Delta : \pi_C \geq \frac{1}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq \frac{1 + x}{2} - \frac{\pi_A}{2} \wedge \pi_C \geq x + \pi_A \right\}
\]

\[
E_{AB} = \left\{ (\pi_A, \pi_C) \in \Delta : \pi_C \geq 1 - 2\pi_A \wedge \pi_C \leq \frac{1 + x}{2} - \frac{\pi_A}{2} \wedge \pi_C \leq 1 + x - 2\pi_A \right\}
\]

Let \( \text{int}(E_{jk}) \) denote the interior of \( E_{jk} \), and \( \text{ext}(E_{jk}) \) its exterior, where \( \text{ext}(E_{jk}) = \Delta \setminus E_{jk} \).

**Proposition 5** Let Assumptions 1–2 hold. Let voter \( i \) be an \( AB \)-type with \( q > 0 \). Then,

\[
\begin{align*}
i) \quad & \quad p^N_{AC} \xrightarrow{N \to \infty} \begin{cases} 1, & \text{if } (\pi_A, \pi_C) \in \text{int}(E_{AC}) \\ 0, & \text{if } (\pi_A, \pi_C) \in \text{ext}(E_{AC}) \end{cases} \\
ii) \quad & \quad p^N_{CB} \xrightarrow{N \to \infty} \begin{cases} 1, & \text{if } (\pi_A, \pi_C) \in \text{int}(E_{CB}) \\ 0, & \text{if } (\pi_A, \pi_C) \in \text{ext}(E_{CB}) \end{cases} \\
n) \quad & \quad p^N_{AB} \xrightarrow{N \to \infty} \begin{cases} 1, & \text{if } (\pi_A, \pi_C) \in \text{int}(E_{AB}) \\ 0, & \text{if } (\pi_A, \pi_C) \in \text{ext}(E_{AB}) \end{cases}
\end{align*}
\]

**Proof.** The proof results straightforwardly from the weak Law of Large Numbers (LLN) and from the inequalities involving \( \tilde{a}, \tilde{b} \) and \( \tilde{c} \), that define each of the probabilities above.

Let \( (\tilde{a}, \tilde{b}, \tilde{c}) \) denote the actual vote profile vector of the other \( \bar{O} \) voters. If \( (\tilde{a}, \tilde{c}) \sim \text{trinomial} \left( \bar{O}, \pi_A, \pi_C \right) \), then the sequence \( (\tilde{a}, \tilde{c}) \) satisfies the weak LLN, so that:

\[
\tilde{a} \sim \bar{O} \rightarrow \pi_A, \quad \tilde{c} \sim \bar{O} \rightarrow \pi_C
\]

\[
\lim_{N \to \infty} P \left( \left| \frac{j}{\bar{O}} - \pi_j \right| < \varepsilon \right) = 1, \forall \varepsilon > 0, \quad j = \tilde{a}, \tilde{b}, \tilde{c}
\]

\[i^) \quad \text{From } \tilde{c} > \tilde{a} \geq \tilde{c} - \tilde{M} > \tilde{b}, \text{ replacing } \tilde{b} = \bar{O} - \tilde{a} - \tilde{c}, \text{ dividing all inequalities by } \bar{O} \text{ and taking limits as } N \to \infty, \text{ we obtain, from the weak LLN that the result must hold whenever}
\]

\[
\pi_C > \pi_A \wedge \pi_C < x + \pi_A \wedge \pi_C > \frac{1 + x}{2} - \frac{\pi_A}{2}
\]
ii) Similarly, from \( \bar{c} > \bar{b} \geq \bar{c} - \bar{M} > \bar{a} \), we obtain that the result must hold whenever
\[
\pi_C > 1 - \frac{\pi_A}{2} \wedge \pi_C < \frac{1 + x}{2} - \frac{\pi_A}{2} \wedge \pi_C > x + \pi_A
\]

iii) Finally, from \((\bar{a} \geq \bar{b} \land \bar{M} + \bar{b} > \bar{a} \land \bar{M} + \bar{b} \geq \bar{c}) \lor (\bar{a} < \bar{b} \land \bar{M} + \bar{a} > \bar{c} \land \bar{M} + \bar{a} > \bar{b})\), we get that the result must hold whenever
\[
\begin{align*}
(\pi_C & > 1 - 2\pi_A \wedge \pi_C < \frac{1 + x}{2} - \frac{\pi_A}{2} \wedge \pi_C < 1 + x - 2\pi_A \\
\lor \pi_C & < 1 - 2\pi_A \wedge \pi_C < x + \pi_A \wedge \pi_C > 1 - x - 2\pi_A)
\end{align*}
\]

Note that the above proposition does not characterize the limits of the above probabilities in all the euclidean zero-measure frontiers of the sets \( E_{jk}, \text{Front}(E_{jk}) \). However, the following proposition allows for a characterization of RBR almost everywhere in the simplex.

Proposition 6 completes proposition 5, in the sense that it states how an \( AB^-\)-type rhizomatically best responds, as a function of \( \pi_A \) and \( \pi_C \), in the regions where all three probabilities, \( p_{AC}^N, p_{CB}^N \) and \( p_{AB}^N \) go to zero as \( N \to \infty \).

Again, with no loss of generality, we state it for an \( AB^-\)-type with \( q > 0 \). Let:

\[
E_1 = \{ (\pi_A, \pi_C) \in \Delta \setminus \{(0, 0)\} : \pi_C \leq \pi_B \wedge \pi_B \geq x + \pi_A \wedge x < 1 \}
\]
\[
E_2 = \{ (\pi_A, \pi_C) \in \Delta \setminus \{(0, 1)\} : \pi_C \geq x + \pi_B \wedge \pi_C \geq x + \pi_A \wedge x < 1 \}
\]
\[
E_3 = \{ (\pi_A, \pi_C) \in \Delta : \pi_C \leq \pi_A \wedge \pi_A \geq x + \pi_B \wedge x < 1 \}
\]

Proposition 6 Let Assumptions 1–2 hold. Let voter \( i \) be an \( AB^-\)-type with \( q > 0 \). Asymptotic RBR (as \( N \to \infty \)) are as follows:

i) If \( q_{AB} \geq \frac{1}{2} \) \( \Leftrightarrow x = \frac{q_{AB}}{1 - q_{AB}} \geq 1 \), an \( AB^-\)-type will vote for A.

ii) Let \( (\pi_A, \pi_C) \in E_1 \). If \( q_{AB} \geq \frac{1}{2} \), an \( AB^-\)-type will vote for A. If \( q_{AB} < \frac{1}{3} \) \( \Leftrightarrow x = \frac{q_{AB}}{1 - q_{AB}} < \frac{1}{2} \), let \( \overline{q}_{AB} \) be given implicitly and uniquely by the equation

\[
\ln \frac{\pi_C + \sqrt{x^2 \pi_C^2 - 4\pi_A \pi_B x^2 + 4\pi_A \pi_B}}{(1 - x^2)(\pi_A + 2\sqrt{x^2 \pi_C^2 - 4\pi_A \pi_B x^2})} = \ln x - \frac{1}{2\pi_B - 2x\pi_B} \left( x\pi_C + \sqrt{x^2 \pi_C^2 - 4\pi_A \pi_B x^2 + 4\pi_A \pi_B} \right)
\]

then, if \( \pi_A > 0 \), an \( AB^-\)-type will vote for B whenever \( q < \overline{q}_{AB} \) and will vote for A whenever \( q > \overline{q}_{AB} \). If \( \pi_A = 0 \), an \( AB^-\)-type will vote for B.

iii) Let \( (\pi_A, \pi_C) \in E_2 \). If \( q_{AB} \geq \frac{1}{2} \), an \( AB^-\)-type will always vote for A. If \( q_{AB} < \frac{1}{2} \), an \( AB^-\)-type will vote for B whenever \( \pi_B > \pi_A \), and will vote for A whenever \( \pi_B < \pi_A \).

iv) Let \( (\pi_A, \pi_C) \in E_3 \). Then, an \( AB^-\)-type will always vote for A.

v) If \((\pi_A, \pi_C) = (0, 0)\) or \((\pi_A, \pi_C) = (0, 1)\), an \( AB^-\)-type will vote for A.

Note that when \( x = 0 \), 4 simplifies to \( \pi_C + 2\sqrt{\pi_A \pi_B} = 1 \) \( \Leftrightarrow \pi_C + 2\sqrt{\pi_A \pi_B} = \pi_A + 2\sqrt{\pi_A \pi_B} \Leftrightarrow \pi_C = \pi_A \), \( \forall (\pi_A, \pi_C) \in E_1 \), and so proposition 6 holds also for \( q = 0 \), in which case it captures the results from Palfrey [26].

The proof of proposition 6 proceeds in several steps and is presented in appendix B. However, since it relies on very different techniques from those used in Palfrey [26], it is instructive to sketch the main steps of the proof in the text.

Sketch of the proof of Proposition 6: Let \( (\pi_A, \pi_C) \in E_1 \cap int(\Delta) \) and \( q_{AB} < \frac{1}{3} \). From 3, an \( AB^-\)-type will vote for A as long as \( \lambda_{AB} < \lim_{N \to \infty} \frac{1 + \frac{p_{AC}^N}{p_{AB}^N}}{1 + \frac{p_{CB}^N}{p_{AB}^N}} \). In order to characterize asymptotic RBR, we need
Figure 1: The diagram represents the partition of the simplex into the relevant regions for propositions 5 and 6.

To compute \( \lim_{N \to \infty} \frac{1}{1 + \frac{p_N}{p_{AB}}} \), in a set in which all three probabilities go to zero as \( N \) grows large. Furthermore, we cannot use the approximation of the trinomial distribution to the bivariate normal distribution, since this approximation is valid only when the deviation of \( j \) from \( N \pi_j \) is of the order of \( \sqrt{N} \), \( j = a, b, c \), and in our setting this deviation is of order \( N \). We are thus in the domain of large deviations/rare events, in which large deviations theory provides us the adequate tools for obtaining asymptotically accurate results. Appendix A contains the definitions and results from large deviations theory that are used in this essay.

Let \( j' = \frac{j}{\sum_j} \), \( j \in \{a, b, c\} \). Then \( a' + b' + c' = 1 \). Let

\[
I(a', c') = a' \ln \frac{a'}{\pi_A} + (1 - a' - c') \ln \frac{1 - a' - c'}{\pi_B} + c' \ln \frac{c'}{\pi_C}
\]

be the (unique) rate function of the trinomial distribution with parameters \( \pi_j \), \( j \in \{A, B, C\} \) (\( b' = 1 - a' - c' \) was used in the above expression, and \( 0 \ln 0 \) is conventionally defined to be 0).

Let also,

\[
\begin{align*}
\epsilon_{AC} &= \{(a', b', c') : c' \geq a' \geq c' - x \geq b', a' + b' + c' = 1, a', b', c' \geq 0\} \\
\epsilon_{CB} &= \{(a', b', c') : c' \geq b' \geq c' - x \geq a', a' + b' + c' = 1, a', b', c' \geq 0\} \\
\epsilon_{AB} &= \{(a', b', c') : x + b' \geq a' \wedge x + b' \geq c' \wedge a' \geq b' \vee x + a' \geq c' \wedge x + a' \geq b' \wedge x + a' \leq b', a' + b' + c' = 1, a', b', c' \geq 0\}
\end{align*}
\]

Working with the rate function of the trinomial distribution and applying Sanov’s Theorem, we establish the following lemma:
Lemma 7 Let Assumptions 1–2 hold. Let voter i be an AB–type with \( q > 0 \). Let \( \pi_j > 0, j = A, B, C \). Let \((\pi_A, \pi_C) \notin e_{jk}\). Then,
\[
\lim_{N \to \infty} \frac{1}{N} \ln p_{jk}^N = -\min_{(a', c') \in e_{jk}} I(a', c')
\]
(6)

Using Stirling’s approximation,
\[
O! \approx \sqrt{2\pi O} \left( \frac{Q}{e} \right)^O
\]

it is straightforward to establish that the probability function
\[
f(a, c) = \frac{O!}{a!(Q - a - c)!cl^a \pi_A^{O-a} \pi_B^a \pi_C^c}
\]

verifies the following identity
\[
\lim_{N \to \infty} \frac{1}{N} \ln f(\ldots) = -I(\ldots).
\]
(7)

Since \( I \) is strictly convex, 6 provides a simple way of computing the rate at which all three probabilities tend to zero, by solving optimization problems with unique optimizers. Sanov’s Theorem gives expression to the intuitive fact that, asymptotically, almost all of the mass of \( p_{jk}^N \) lies in the "closest" point in \( e_{jk} \), as measured by the statistical distance \( I(\ldots) \), to \((\pi_A, \pi_C) \in E_1 \cap \text{int}(\Delta)\). Thus, unless \( \min_{(a', c') \in e_{AB}} I(a', c') = \min_{(a', c') \in e_{AC}} I(a', c') \), which happens when 4 holds, \( p_{CB}^N \) and \( p_{AB}^N \) will converge to zero at different rates. More precisely, if \( \bar{q} \) is given by 4, we have
\[
\lim_{N \to \infty} \frac{p_{CB}^N}{p_{AB}^N} = \begin{cases} \infty, q < \bar{q} \\ 0, q > \bar{q} \end{cases}
\]

Noting that \( \lim_{N \to \infty} \frac{p_{AC}^N}{p_{AB}^N} = 0 \), since any point in \( e_{AC} \) is more "distant" that any point in \( e_{AB} \), for all \((\pi_A, \pi_C) \in E_1 \cap \text{int}(\Delta)\), we obtain the cutoff \( \bar{x}_{AB} \) is such that
\[
\bar{x}_{AB} = \lim_{N \to \infty} \frac{1 + \frac{p_{AC}^N}{p_{AB}^N}}{1 + \frac{p_{CB}^N}{p_{AB}^N}} = \begin{cases} \infty, q < \bar{q} \\ 0, q > \bar{q} \end{cases}
\]

Hence, an AB–type will vote for A if \( q > \bar{q} \) and for B if \( q < \bar{q} \).

For \( q_{AB} \geq \frac{1}{2} \), it is easy to establish that an AB–type always votes for A (see appendix B). Finally, the validity of the result is straightforwardly established for the cases where \( \pi_A = 0 \) or \( \pi_C = 0 \).

For \((\pi_A, \pi_C) \in E_2\) the steps are identical, only now the solution of \( \min_{(a', c') \in e_{CB}} I(a', c') = \min_{(a', c') \in e_{AC}} I(a', c') \) is simply \( \pi_B = \pi_A \). Also, the cutoff \( \bar{x}_{AB} \) is:
\[
\bar{x}_{AB} = \lim_{N \to \infty} \frac{p_{AB}^N}{p_{AC}^N + p_{CB}^N} = \begin{cases} \infty, \pi_A > \pi_B \\ 0, \pi_A < \pi_B \end{cases}
\]

Finally, if \((\pi_A, \pi_C) \in E_3\), it is immediate that if an AB–type votes for A, A always wins. This ends the sketch of the proof.

Gathering propositions 5 and 6 immediately establishes our main result for RBR.

Let \( E = \{(\pi_A, \pi_C) \in \Delta : \pi_C < 1 - 2\pi_A \wedge \pi_C > x + \pi_A\} \cup \{(\pi_A, \pi_C) \in E_1 : q < \bar{q}_{AB}\} \), where \( \bar{q}_{AB} \) is as defined in proposition 6.

---

11The assumption that \((\pi_A, \pi_C) \notin e_{jk}\) is unnecessary: however, the previous proposition already establishes that in the case in which \((\pi_A, \pi_C) \notin \text{int}(e_{jk})\), \( \lim_{N \to \infty} \frac{1}{N} \ln p_{jk}^N = 0 \).
Theorem 8 Let Assumptions 1–2 hold. Let voter $i$ be an AB-type with $q > 0$. Asymptotic RBR (as $N \to \infty$) are as follows:

i) Let $(\pi_A, \pi_C) \in E$. Then, an AB-type will vote for $B$ a.e.\textsuperscript{12}

ii) Let $(\pi_A, \pi_C) \in \Delta/\text{ext}(E)$. Then, an AB-type will vote for $A$.

Figure 2.2 below illustrates RBR as characterized by the theorem.

The theorem above ensures that, in particular, an AB-type never votes for $B$ whenever $\pi_A < \min \{\pi_B, \pi_C\}$, which is a feature that our model shares with Palfrey’s, as it intuitively should. Let $\pi_A > \min \{\pi_B, \pi_C\}$ and in specific, $\pi_A < \pi_B \leq \pi_C$, with no loss of generality. The cutoffs for $\lambda_{Aj}$, $j \in \{B,C\}$, are always either 0 or greater than or equal to 1, but $\lambda_{Aj}$ is a function of $q_{Aj}$, parametrized by $\bar{q}_{Aj}$\textsuperscript{13}; the equations that confirm equilibrium probabilities therefore depend only on $G_{Aj}(\bar{q}_{Aj})$ and are given in our second main technical result.

Let $\bar{q}_{AC} = \frac{\pi_{AC}}{\pi_{AC}(1+\pi_{AC})}$, with $\pi_{AC} \in [0,1)$, be (uniquely) determined by

$$\ln \frac{\pi_B + \sqrt{\pi_{AC}^2 \pi_B^2 - 4 \pi_A \pi_C \pi_{AC}^2 + 4 \pi_A \pi_C}}{(1 - \pi_{AC}^2) \left(\pi_A + 2 \sqrt{\pi_B \pi_C}\right)} = \pi_{AC} \ln \frac{1}{2 \pi_C - 2 \pi_{AC} \pi_C} \left(\pi_{AC} \pi_B + \sqrt{\pi_{AC}^2 \pi_B^2 - 4 \pi_A \pi_C \pi_{AC}^2 + 4 \pi_A \pi_C}\right)$$

(8)

Theorem 9 Let Assumptions 1–2 hold. Let $(\pi_A, \pi_B, \pi_C)$ be the equilibrium probabilities of a RNE, such that $\pi_A < \pi_B \leq \pi_C$. Let $\bar{q}_{AC}$ be given uniquely by 8 if $\pi_A > 0$ and $\bar{q}_{AC} = \frac{1}{3\pi_{AC}}$ if $\pi_A = 0$. Then,

\textsuperscript{12}RBRs are completely characterized in proposition 6, with the exception of the curves that give the cutoffs $\bar{q}_{AB}$.

\textsuperscript{13}Note that all the results hold even if cutoffs are greater than 1.
\[
\begin{align*}
\pi_A &= l_{AB} (1 - G_{AB} (\frac{\pi_C - \pi_A}{l_{AB} (1 + \pi_C - \pi_A)})) + l_{AC} (1 - G_{AC} (\overline{q}_{AC})) \\
\pi_B &= l_{BA} + l_{BC} G_{AB} (\frac{\pi_C - \pi_A}{l_{AB} (1 + \pi_C - \pi_A)}) \\
\pi_C &= l_{CA} + l_{CB} + l_{AC} G_{AC} (\overline{q}_{AC})
\end{align*}
\]

Proof. Let \( E' = \{ (\pi_A, \pi_C) \in \Delta : \pi_A < \pi_B \leq \pi_C \} \). From propositions 5 and 6, an \( AB \)-type votes \( B \) if \( (\pi_A, \pi_C) \in E' \cap (E_{CB} \cup E_2) \) and \( q_{AB} l_{AB} < \frac{1}{2} \)
and votes \( A \) if \( (\pi_A, \pi_C) \in E_{AB} \) or \( q_{AB} l_{AB} \geq \frac{1}{2} \). Thus, an \( AB \)-type votes \( A \) if \( x_{AB} > \pi_C - \pi_A \Leftrightarrow q_{AB} > \frac{\pi_C - \pi_A}{l_{AB} (1 + \pi_C - \pi_A)} \) and votes \( B \) if \( q_{AB} < \frac{\pi_C - \pi_A}{l_{AB} (1 + \pi_C - \pi_A)} \).

The results for \( q_{AC} \) obtain directly from proposition 6, given the symmetry of the results for \( q_{AB} \) and \( q_{AC} \) in the exchange of \( \pi_B \) for \( \pi_C \) and vice-versa. \( \blacksquare \)

On one hand, we can check from this characterization of the equilibria that strategic voting for the second best candidate will be (weakly) decreasing with respect to rhizomatic beliefs. In rough terms, a more rhizomatic agent will tend to vote more often for her preferred candidate.

On the other hand, this theorem will allow us to calculate the set of possible equilibria for each polity. In fact, an equilibrium always exists.

Lemma 10 Under Assumptions 1 and 2, there is at least one Duvergerian RNE for each polity.

Proof. It is straightforward to check that, under Assumption 1, there are at least four types \( jk \) such that \( j, k \in \{A, B, C\} : j \neq k \) for which \( l_{jk} < \frac{1}{3} \)^{14}

^{14} If Assumption 1 were relaxed, we could have polities for which only three types would satisfy \( l_{jk} < \frac{1}{3} \). However, the result would still hold a.e.
In particular, there must be at least one \( j \) such that \( l_{kj} < \frac{1}{3} \) and \( l_{jj} < \frac{1}{3} \). Without loss of generality, let \( l_{AB} < \frac{1}{3} \) and \( l_{AC} < \frac{1}{3} \).

There are three possible Duvergerian outcomes. Again without loss of generality, let \( l_{BA} + l_{BC} + l_{AB} < l_{CA} + l_{CB} + l_{AC} \).

The Duvergerian outcome where \( A \) loses would then be such that \( \pi_A < \pi_B \leq \pi_C \), \( \pi_A = 0 \) and \( \pi_C > \frac{1}{2} \). We have that \( \bar{q}_{AC} = \frac{1}{3} \) and therefore \( G_{AC}(\bar{q}_{AC}) = 1 \). Moreover, \( l_{AB}(1+\pi_B-\pi_A) = l_{AB}(1+\pi_C) > 1 \Leftrightarrow \pi_C > \frac{l_{AB}}{1-l_{AB}} \) since \( \frac{l_{AB}}{1-l_{AB}} < \frac{1}{2} < \pi_C \).

Therefore, \( G_{AB}(\frac{l_{AB}+l_{AC}}{l_{AB}(1+\pi_C-\pi_A)}) = 1 \). But then from 9 we have \( \pi_A = 0, \pi_B = l_{BA} + l_{BC} + l_{AB} \) and \( \pi_C = l_{CA} + l_{CB} + l_{AC} \) and the equilibrium holds. \( \blacksquare \)

We therefore conclude that at least one Duvergerian equilibrium will always remain in this setting. On one hand, this seems to reinforce Palfrey’s [26] conclusion. On the other hand, Palfrey concluded that there would always be three two-party equilibria such that each party would lose in exactly one of them regardless of the preferences. Our next step is to investigate whether equilibrium selection is possible in our model and whether non-Duvergerian outcomes can also arise.

Notice that a possible computational drawback of our model might result from the fact that 8 does not have a closed form solution for threshold \( \bar{q}_{AC} \). However, numerical computations need not allow for general distributions of RB. Since RNE conditions depend only on \( G_{jk}(\bar{q}_{jk}), j = A, B, C \), the properties of RNE can be analyzed, with no loss of generality, using only simple discrete distributions \( H_{jk}(\cdot) \) for which

\[
H_{jk}(q_{jk}) = \begin{cases} 
   p_{jk}, & q_{jk} = 0 \\
   1 - p_{jk}, & q_{jk} = 1 
\end{cases} \tag{10}
\]

This procedure brings about two main advantages: the simplification of RNE computations, since equations 9 allow for a straightforward check of RNE conditions; and most importantly, an insightful analysis of implications of possible correlations among types and beliefs.

After equilibrium probabilities are computed, together with values for the relevant cutoffs, these will remain the equilibria for any distribution that leads to the same accumulated probability mass at each cutoff.

Using such discrete distributions, lemma 10 has an appealing partial converse: the following result provides sufficient conditions for a given candidate say, \( C \) to be the unique winner of the election, given that all her supporters are rhizomatic enough. From lemma 10, this can only be possible if either \( l_{CA} > \frac{1}{3} \) or \( l_{CB} > \frac{1}{3} \).

**Lemma 11** Let Assumptions 1–2 hold. Let \( p_{CA} = p_{CB} = 0 \). Then, if \( l_C = l_{CA} + l_{CB} \geq \frac{1}{2} \) and \( \max\{l_{CA}, l_{CB}\} > \frac{1}{3} \), \( C \) always wins the election.

**Proof.** Without loss of generality, the proof is given for \( l_{CA} > \frac{1}{3} \).

\( C \) cannot be the second most voted-for candidate: in such a case, theorem 10 ensures that \( CA- \) and \( CB- \)types always vote for \( C \), which implies that \( \pi_C \geq l_C \geq \frac{1}{2} \) and \( \max\{\pi_A, \pi_B\} \leq \max\{l_A + l_{BA}, l_B + l_{AB}\} < \frac{1}{2} \leq \pi_C \).

Suppose then that \( \pi_C = 0 < \pi_B \leq \pi_A \) in any equilibrium with \( p_{CA} = p_{CB} = 0 \): that is not possible, since \( \bar{q}_{CA} = \frac{1}{3} \) and then \( CA- \)types would vote for \( C \).

Suppose that \( \pi_C = 0 < \pi_A < \pi_B \) in any equilibrium with \( p_{CA} = p_{CB} = 0 \): if \( l_{CB} > \frac{1}{3} \), then \( \bar{q}_{CB} = \frac{1}{3} \) and \( CB- \)types would vote for \( C \), breaking any such Duvergerian equilibrium; if \( l_{CB} < \frac{1}{3} \), then \( \bar{q}_{CB} = \frac{1}{3} \) and \( CB- \)types vote for \( B \).

For any such Duvergerian equilibrium to hold, it is necessary that \( \frac{\pi_B}{l_{CA}(1+\pi_B)} = \frac{1-l_{A}+l_{CA}}{l_{CA}(2-l_{A}-l_{CA})} \geq 1 \Leftrightarrow l_{A} \leq \frac{1-3l_{CA}-l_{CA}^2}{1-l_{CA}} \). But \( l_{CA} > \frac{1}{3} \Rightarrow 1-3l_{CA}-l_{CA}^2 < -\frac{1}{5} \Rightarrow l_{A} < 0 \), which is not possible, and the equilibrium breaks.

Thus, there are no Duvergerian equilibria in which \( \pi_C = 0 \leq \min\{\pi_A, \pi_B\} \) in case \( p_{CA} = p_{CB} = 0 \).

\[\text{We again ignore the knife-edge cases where } l_{CA} = \frac{1}{3} \text{ or } l_{CB} = \frac{1}{3}.\]
Also, there are no three-party equilibria in which \( \pi_C = l_{CA} < \min \{ \pi_A, \pi_B \} \): if \( \pi_A = l_A > l_{CA} \), then \( \pi_B = 1 - l_A - l_{CA} < \frac{1}{3} < \pi_C \); if \( \pi_B = 1 - l_A - l_{CA} > \pi_C = l_{CA} \), then \( \pi_A = l_A < 1 - 2l_{CA} < \frac{1}{3} < l_{CA} = \pi_C \).

The only possibility is that \( \pi_C = l_{CB} < \min \{ \pi_A, \pi_B \} \). Then, \( \pi_A = l_A + l_{CA} \) and \( \pi_B = l_B \).

Suppose that \( \pi_C = l_{CB} < \pi_A = l_A + l_{CA} \leq \pi_B = l_B \). This requires that
\[
\frac{\pi_B - \pi_C}{l_{CA}(1 + \pi_B - \pi_C)} \geq 1 \Leftrightarrow \\
l_A(1 - l_{CA}) \leq (1 - 2l_C)(1 - l_{CA}) - l_{CA}^2 \Rightarrow l_A < 0, \text{ which is impossible with } p_{CA} = p_{CB} = 0, \text{ breaking this possible equilibrium outcome.}
\]

Suppose that \( \pi_C = l_{CB} < \pi_B = l_B \leq \pi_A = l_A + l_{CA} \). This cannot be an equilibrium outcome with \( p_{CA} = p_{CB} = 0 \), since \( \bar{p}_{CA} \) given by 8 is always lower than \( \frac{1}{3} < \frac{1}{3} \) and \( CA \)-types would vote for \( C \).

That \( l_C \geq \frac{1}{2} \) is a necessary condition for the result to hold is established by the following counter-example. Let \( l_C < \frac{1}{2} \), then \( t_B = l_{BA} \) and \( t_A = l_{AB}, p_{BA} = 1 \) and \( p_{AB} = 0 \).

Then, \( \pi_B = 0 < \pi_C = l_C < \pi_A = l_A + l_{BA} \) is obviously a possible Duvergerian outcome. \( \blacksquare \)

Lemma 11 provides a first important implication of our model: if there is a candidate who is preferred by at least half of the voters, and more than \( \frac{1}{3} \) of those voters have the same preference ordering, then the Condorcet winner (CW) will be selected as long as the agents are rhizomatic enough. Whereas Palfrey [26] concludes that any candidate can lose (with zero votes) in a large election under Plurality Rule, our model in fact selects the CW as the only possible winner in such a case.

## 4 Examples and applications

This section analyzes RNE properties for some pertinent examples and draws general insights from our model.

### Example 1

Let \( p_{jk} = 0 \) for all \( j, k \in \{A, B, C\} : j \neq k \). Consider the following polity:

<table>
<thead>
<tr>
<th>( l_{AB} )</th>
<th>( l_{AC} )</th>
<th>( l_{BA} )</th>
<th>( l_{BC} )</th>
<th>( l_{CA} )</th>
<th>( l_{CB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.14</td>
<td>0.4</td>
<td>0</td>
<td>0.45</td>
<td>0</td>
</tr>
</tbody>
</table>

In this case, preferences are single-peaked and we have a unique equilibrium characterized by:

\[
\begin{align*}
\pi_A & = 0 \\
\pi_B & = 0.41 \text{ and } \bar{p}_{AC} \simeq 2.381 \text{ and } \frac{\pi_C - \pi_A}{l_{AB}(1 + \pi_C - \pi_A)} \simeq 37.107. \\
\pi_C & = 0.59
\end{align*}
\]

On one hand, this example suffices to show that equilibrium selection can arise depending on the preferences and rhizomatic beliefs. On the other hand, the example shows us that a Condorcet winner may not win in the unique equilibrium prediction: in fact, \( A \) would be the Condorcet winner in this case.

### Example 2

Let \( p_{jk} = 0 \) for all \( j, k \in \{A, B, C\} : j \neq k \). Consider the following polity:

<table>
<thead>
<tr>
<th>( l_{AB} )</th>
<th>( l_{AC} )</th>
<th>( l_{BA} )</th>
<th>( l_{BC} )</th>
<th>( l_{CA} )</th>
<th>( l_{CB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.18</td>
<td>0.3</td>
<td>0.01</td>
<td>0.35</td>
<td>0.01</td>
</tr>
</tbody>
</table>

In this case, we have three equilibria, one of which is not Duvergerian:

\[\text{\footnote{It is a consequence of proposition 6 that } \bar{p}_{CA} \text{ given by 8 is lower than } \frac{1}{3}, \text{ since the cutoff when } \pi_C > 0 \text{ is always lower than if } \pi_C = 0.}\]

18
\[
\begin{align*}
\pi_A &= 0.33 \\
\pi_B &= 0.3 \\
\pi_C &= 0.37
\end{align*}
\] and \( q_{BC} \simeq 1.501 \) and \( \frac{\pi_C - \pi_B}{\theta_B(1 + \pi_C - \pi_B)} \simeq 0.218 \).

We can conclude that even though at least one equilibrium satisfies Duverger’s Law, there may be equilibria where Duverger’s Law may fail to hold. In this case, \( BA \)-types vote \( B \) because they are rhizomatic; otherwise, they would vote \( A \) and lead to the victory of the Condorcet winner.

**Example 3**

Let \( p_{jk} = 0 \) for all \( j, k \in \{ A, B, C \} : j \neq k \). Consider the following polity:

<table>
<thead>
<tr>
<th>( l_{AB} )</th>
<th>( l_{AC} )</th>
<th>( l_{BA} )</th>
<th>( l_{BC} )</th>
<th>( l_{CA} )</th>
<th>( l_{CB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.04</td>
<td>0.18</td>
<td>0.1</td>
<td>0.21</td>
<td>0.34</td>
<td>0.13</td>
</tr>
</tbody>
</table>

In this case, we have two possible equilibria, one of which is not Duvergerian:

\[
\begin{align*}
\pi_A &= 0.18 \\
\pi_B &= 0.35 \\
\pi_C &= 0.47
\end{align*}
\] and \( q_{AC} \simeq 0.780 \) and \( \frac{\pi_C - \pi_A}{\theta_B(1 + \pi_C - \pi_A)} \simeq 5.620 \) and

\[
\begin{align*}
\pi_A &= 0 \\
\pi_B &= 0.35 \\
\pi_C &= 0.65
\end{align*}
\] and \( q_{AC} \simeq 1.852 \) and \( \frac{\pi_C - \pi_A}{\theta_B(1 + \pi_C - \pi_A)} \simeq 9.848 \).

In this case, we can confirm that the model allows for equilibrium selection and for non-Duvergerian outcomes. Moreover, even though uniqueness of equilibrium is lost, we do have a unique prediction with respect to the ordering of candidates.

**Example 4**

Let \( p_{CA} = p_{CB} = 0 \) and let \( p_{jk} = 1 \) for \( j \in \{ A, B \}, k \in \{ A, B, C \} : k \neq j \).

Consider the following polity:

<table>
<thead>
<tr>
<th>( l_{AB} )</th>
<th>( l_{AC} )</th>
<th>( l_{BA} )</th>
<th>( l_{BC} )</th>
<th>( l_{CA} )</th>
<th>( l_{CB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.28</td>
<td>0.06</td>
<td>0.2</td>
<td>0.01</td>
<td>0.35</td>
</tr>
</tbody>
</table>

In this case, we have two Duvergerian equilibria:

\[
\begin{align*}
\pi_A &= 0 \\
\pi_B &= 0.36 \\
\pi_C &= 0.64
\end{align*}
\] and \( q_{AC} \simeq 1.190 \) and \( \frac{\pi_C - \pi_A}{\theta_B(1 + \pi_C - \pi_A)} \simeq 3.902 \) and

\[
\begin{align*}
\pi_A &= 0.44 \\
\pi_B &= 0 \\
\pi_C &= 0.56
\end{align*}
\] and \( q_{BC} \simeq 1.667 \) and \( \frac{\pi_C - \pi_B}{\theta_B(1 + \pi_C - \pi_B)} \simeq 5.983 \).

This example is particularly informative with respect to the importance of the correlations between preferences and beliefs. If \( CA \)-types and \( CB \)-types were not rhizomatic, any Duvergerian equilibrium might arise. The equilibrium where \( C \) would be defeated is ruled out due to the rhizomatic beliefs. Whereas agents who prefer candidates \( A \) or \( B \) are easily willing to vote for their second preference, that is not the case with voters who prefer \( C \). Therefore, it is possible for \( C \) to ensure the win by emphasizing group identity and stimulating rhizomatic thinking. In many instances, this is the type of behavior that can be observed in campaigns that call for strategic voting.

It is worth remarking that a reversal of \( l_{AB} \) and \( l_{AC} \) would lead the first equilibrium to disappear and be replaced by an equilibrium where:
Thus, in this case, a strong feeling of group identity would benefit...a majority of...leading to the victory of...equilibrium outcome in which...RB that attributes a mass...yields a unique equilibrium outcome winner, even though...win and...win and...polities, a small proportion of rhizomatic partisans of a given candidate may still be enough to ensure her...specific distributions of types in a more intricate way. The next example shows that, however, for pertinent...both rhizomatic and non-rhizomatic agents. Whereas possible violations of Duverger’s Law can clearly carry...outcomes. In more realistic settings, however, we would expect more balanced distributions of RB, with...to better understand the effects that the correlations of the distributions of types have on equilibrium...These are the Duvergerian equilibrium outcomes predicted by Palfrey [26].

\[
\left\{ \begin{array}{l}
\pi_A = 0 \\
\pi_B = 0.54 \quad \text{and} \quad \eta_{AB} \simeq 1.190 \quad \text{and} \quad \frac{\pi_B - \pi_A}{\eta_{AC}(1 + \pi_B - \pi_A)} \simeq 3.506.
\end{array} \right.
\]

The main conclusion we can derive from this example is that the distribution of preferences is also crucial to realize whether political strategies should strive to foster group identity around a given candidate.

The assumption that the mass of rhizomatic agents in a given class is either 0 or 1 has helped us to better understand the effects that the correlations of the distributions of types have on equilibrium outcomes. In more realistic settings, however, we would expect more balanced distributions of RB, with both rhizomatic and non-rhizomatic agents. Whereas possible violations of Duverger’s Law can clearly carry through just the same, potential selection of uniquely determined equilibrium outcomes will depend on the specific distributions of types in a more intricate way. The next example shows that, however, for pertinent polities, a small proportion of rhizomatic partisans of a given candidate may still be enough to ensure her victory.

**Example 5**

Let \( p = p_{CA} \in (0, 1) \) and let \( p_{jk} = 1 \) for \( j \in \{A, B\}, k \in \{A, B, C\} : k \neq j \).

Consider the following polity:

<table>
<thead>
<tr>
<th>( t_{AB} )</th>
<th>( t_{AC} )</th>
<th>( t_{BA} )</th>
<th>( t_{BC} )</th>
<th>( t_{CA} )</th>
<th>( t_{CB} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.22</td>
<td>0.13</td>
<td>0.14</td>
<td>0.03</td>
<td>0.48</td>
<td>0</td>
</tr>
</tbody>
</table>

The Duvergerian equilibrium outcomes if all voters are non-rhizomatic i.e. if \( p_{jk} = 1 \), for \( j, k \in \{A, B, C\} : k \neq j \), would be\(^{17}\):

\[
\left\{ \begin{array}{l}
\pi_A = 0 < \pi_B = 0.39 < \pi_C = 0.61 \\
\pi_B = 0 < \pi_A = 0.49 < \pi_C = 0.51 \\
\pi_C = 0 < \pi_B = 0.17 < \pi_A = 0.83
\end{array} \right.
\]

and so \( C \) would never be the second most voted-for candidate and \( B \) could never win. Furthermore, \( C \) is the Condorcet winner and \( B \) the Condorcet loser.

But the equilibrium outcome \( \pi_C < \pi_B < \pi_A \) can be eliminated under mild conditions regarding \( p_{CA} \), leading to the victory of \( C \) in all instances.

Note that \( CA\)-types with \( q_{CA} = 1 > \eta_{CA} = \frac{1}{\eta_{CA}} = \frac{1}{3.638} \simeq 0.694 \) will vote for \( C \). If \( B \) comes in last, then \( BA\)-types will transfer their votes to \( A \) and \( BC\)-types to \( C \). \( C \) would therefore win and \( \pi_B = 0 < \pi_A = 0.49 < \pi_C = 0.51 \).

If \( A \) comes in last, then \( AB\)-types will transfer their votes to \( B \) and \( AC\)-types to \( C \). \( C \) would therefore win and \( \pi_A = 0 < \pi_B = 0.39 < \pi_C = 0.61 \).

Thus, for equilibrium outcomes in which \( \pi_C < \pi_B < \pi_A \) to be broken, it is enough that \( 0.48p > \min \{ \pi_B = 0.17, \pi_A = 0.35 + 0.48(1 - p) \} \Leftrightarrow p > \frac{17}{38} \simeq 0.354 \). And if that is the case, \( C \) wins.

This example is illustrative in three main important respects. First, it ensures that \( C \) always wins, yielding a unique equilibrium outcome winner, even though \( C \) is not even preferred by a simple majority, as long as \( 36\% \) of \( C \)'s partisans are rhizomatic enough. This ensures also, in this example, the victory of the Condorcet winner, thus precluding the possible Duvergerian outcome of a massive victory of \( A \). Second, this result holds even if the majority of \( CA\)-types are fully non-rhizomatic, as long as \( 36\% \) are rhizomatic enough. Thus, in this case, a strong feeling of group identity would benefit \( C \)'s partisans. Third, any distribution of \( RB \) that attributes a mass \( p > 0 \) to \( C \)'s partisans with \( q_{CA} \geq 0.7 \) will give rise to a possible non-Duvergerian equilibrium outcome in which \( C \) loses, but gets a non-zero proportion of votes, thus violating Duverger’s Law.

\(^{17}\)These are the Duvergerian equilibrium outcomes predicted by Palfrey [26].
Example 6: The 1992 US Presidential Election

In the 1992 US presidential election, Bill Clinton \( (C) \) received 43% of the popular vote, George Bush \( (B) \) 37.4%, and Ross Perot \( (P) \) 18.9%, in a total of 99.3%.

The US president is elected with at least 270 votes from the Electoral College and not on a simple PR basis, which may have repercussions on strategic voting behavior. Still, in 1992 the Electoral College elected representatives in each state on a winner-take-all simple PR basis. Furthermore, inspection of the popular vote in each state reveals that Duverger’s Law failed to hold in the large majority of states. In Iowa, for instance, the results were almost an exact match of the nationwide popular vote: 43.3% of the votes for Clinton, 37.3% for Bush and 18.7% for Perot in a total of 99.3%.

For simplicity, and since qualitative results do not depend on the exact equilibrium probabilities, we use nationwide results in what follows. The purpose of this exercise is to analyze how RB might help explain the observed equilibrium and identify the determinants for such voting behavior.

Note that, in empirical cases, equilibrium probabilities are known instead of the distribution of preferences: we therefore have to assume some reasonable distribution of preferences, compatible with the outcome of the election. Furthermore, and since there were other candidates receiving votes, an exact analysis would require the use of a multinomial model with more than three alternatives: for this discussion, we normalize equilibrium probabilities to add up to 1 in the trinomial model. Let then \( \pi_C = 0.433 \), \( \pi_B = 0.377 \), and \( \pi_P = 0.190 \) denote the equilibrium probabilities.

It has been argued that Perot may have been partly responsible for Bush’s defeat. The claim is that a majority of \( PB\)–types still voted for Perot, whereas the majority of \( PC\)–types shifted their vote to Clinton. Applying the previous results to these equilibrium probabilities, theorem 9 yields \[ \frac{\pi_C - \pi_P}{\pi_P (1 + \pi_C - \pi_P)} = \frac{0.433 - 0.19}{0.377} = 0.963. \]

and from \( \pi_{BC} = 0.19602 \) \( \Leftrightarrow \pi_{PC} = 0.16389 \). Hypothesize, for instance, that \( l_{BP} + l_{CB} = 0.36 \), \( l_{CP} + l_{PC} = 0.34 \), \( l_{PB} = 0.207 \) and \( l_{PC} = 0.093 \). Then, all \( PC\)–types have \( q_{PC} < \frac{0.16389}{0.377} = 0.433 \). Also, \( \pi_B = l_B + \alpha l_{PB} = 0.36 + 0.207\alpha = 0.377 \Leftrightarrow \alpha = 0.082 \).

Since \( \frac{\pi_C - \pi_P}{\pi_P (1 + \pi_C - \pi_P)} = 0.94444 \), as long as \( G_{PB}(0.94444) = 0.082 \), the equilibrium is sustained.

Note that, clearly, under our assumptions, a Duvergerian outcome would never have given the victory to Clinton, since \( l_{BP} + l_{BC} + l_{PB} = 0.36 + 0.207 = 0.567 > \frac{94444}{0.94444} = 0.34 \), \( l_{PB} + l_{PC} + l_{BP} = 0.66 > \frac{0.34}{0.36} = 0.34 \). Also, under sincere voting, Clinton would also have lost to Bush: \( l_C = 0.34 < l_B = 0.36 \).

Another interesting feature of the hypothesized distribution of preferences is that Perot’s supporters who prefer Clinton in second place are less than half of those that prefer Bush in second place. The key is that they are just enough voters to make Clinton win, but also few enough voters such that they vote Clinton no matter how rhizomatic they may be. In fact, if this number were large enough, rhizomatic \( PC\)–types might prefer to vote Perot, breaking the equilibrium. It is worth stressing that what allows Clinton to win in this scenario is not just the fact that there are just enough \( PB\)–types but also that most of them are very rhizomatic.

Remembering that RT captures the notion of group identity, and assuming the hypothesized distribution of preferences to hold approximately, the outcome of the 1992 US presidential election might be at least partially explained by the following rationale: most of Clinton’s and Bush’s supporters ranked Perot second and had a strong feeling of group identity. A majority of Perot’s supporters preferred Bush to Clinton, again with a strong feeling of group identity. The number of Perot’s supporters preferring Clinton to Bush was smaller but nonetheless high enough to make them anticipate they were more likely to be pivotal for Clinton than for Perot. In that case, \( PC\)–types should shift their vote to Clinton and \( PB\)–types stick with Perot. And this may indeed be part of the story of the 1992 US presidential elections.

We can conclude from these examples that:

i) For given distributions of types, there is equilibrium selection, in the sense that some of Palfrey’s two-party equilibria are eliminated. The equilibrium may be unique (and there are also cases in which the equilibrium winner can be uniquely determined even if uniqueness of equilibrium is lost); namely, whenever at least half of the population prefer a given candidate, and more than one-third of these voters share the same preferences, then their preferred candidate wins the election as long as those voters are sufficiently rhizomatic.

ii) Duverger’s Law may break and that can help explain empirically observed equilibrium outcomes;
Correlations between preferences and beliefs are the key determinants of equilibria. In any case, the diversity of possible equilibrium outcomes that may result from combining different distributions of preferences and RB requires further analysis.

5 Conclusions

We introduce rhizomatic thinking in a pivotal-voter model under Plurality Rule. This is to the best of our knowledge the first time an alternative to the assumption of purely individual rationality is introduced in pivotal-agent games. Our rhizomatic assumption allows any agent to have an exogenous rhizomatic belief \( q_i \) as to the proportion of types with the same preference ranking that \( i \) believes will take the same action as she does.\(^{18}\) Among several other behavioral assumptions, RT captures the feeling of group identity, and provides a horizontal notion of rationality capable of encompassing individual as well as more collective notions of rationality. This assumption impacts on strategic behavior, since now an agent may perceive her pivot probabilities to be non-vanishing as the number of voters grows large. We find that caeteris paribus vote manipulation is (weakly) decreasing in rhizomatic beliefs.

In our model, equilibria always exist and can be determined for any distribution of preferences and rhizomatic beliefs. In particular, unique equilibrium outcomes result for certain distributions of preferences and types. Namely, whenever at least half of the population prefer a given candidate, and more than one-third of these voters share the same preferences, then that candidate will always be selected if that segment of the population is fully rhizomatic. This conclusion eliminates Duvergerian equilibria in which such an alternative is defeated.

A key result of our model is that Duverger’s Law is violated: there are polities with non-Duvergerian equilibrium outcomes. This result cannot be derived from existing models in which individual rationality is not relaxed but it allows us, for instance, to provide a rationale for the three-party equilibrium outcome of the 1992 US presidential election.

Our model also suggests that there is room for an in-depth analysis of correlations between preferences and rhizomatic beliefs, and that this analysis may prove to be useful for political strategists who may or may not wish to enhance feelings of group identity and to call for strategic voting.

Other directions for future research include a global games approach that would allow for uncertainty with respect to each agent’s own rhizomatic belief and own class of like-minded others, leading RBR to have a statistical nature rather than a deterministic one.

Also, the methodology we present for the determination of asymptotic RNE can be generalized to any number of candidates, provided the multinomial distribution is used.

More importantly, the results presented for PR can be both positively and normatively compared with those for other aggregation rules such as Approval Voting (AV). Palfrey argues that the implications of strategic voting are not well understood for AV. Although this claim was made before the contributions of Roger Myerson towards a better understanding of the implications of strategic voting under AV in the Poisson-Myerson environment, it is still true that there is no direct comparison in the literature between PR and AV in a multinomial setting. In a companion paper, we provide such a comparison for both individually rational and rhizomatic agents.

Appendix A - On Large Deviations

We first present the relevant definitions and facts, and state how Sanov’s Theorem reads in our setting.

The reader can find omitted details in den Hollander [16] or in Ellis [11], for example.

There are many possible alternative formulations of Sanov’s Theorem.

We present a simple version, appropriate for our setting, which combines Theorem 3.4. and Corollary 3.5. in Ellis [11].

Let there be \( N \) voters and the number of an \( AB \)-type voter \( i \)'s "like-minded" others be perceived to be \( M \) and \( O = N - M \).

\(^{18}\) We assumed the class of like-minded others to be the class of agents with the same preference ranking.
Let the number of $O$ other voters that vote for candidate $J$ be $j_N$, $j \in \{a, b, c\}$, $a_N + b_N + c_N = O$.

Let the probability mass function (pmf) of $(a_N, c_N)$ be given by the pmf of the trinomial distribution,
\[
f(a_N, c_N) = \frac{O!}{a_N!(O-a_N-c_N)!c_N!} \pi_A^{a_N} \pi_B^{(O-a_N-c_N)} \pi_C^{c_N}.
\]

Let $j_O$ denote the normalized empirical measure, where $j_O = \frac{jN}{O}$.

Let $\Delta$ denote the two-dimensional simplex.

Define
\[
I(a', c') = \begin{cases}
    a' \ln \frac{a'}{\pi_A} + (1 - a' - c') \ln \left(1 - \frac{a' - c'}{\pi_B}\right) + c' \ln \frac{c'}{\pi_C}, & \text{if } (a', c') \in \Delta \\
    +\infty, & \text{o.w.}
\end{cases}
\]

be the unique (strictly convex, lower semi-continuous with compact level sets, and continuously differentiable for $(a', c') \in \text{int}(\Delta))$ rate function of the trinomial distribution, with parameters $\pi_j$, where $0 \ln 0$ is conventionally defined to be 0.

Let $A$ be any Borel subset of $\Delta$. Let $\text{cl}(A)$ denote the closure of $A$ and let $A$ be such that $\text{cl}(\text{int}(A)) = \text{cl}(A)$.

Under the above conditions, Sanov’s Theorem reads as follows.

**Sanov’s Theorem:** Let $A$ be any Borel subset of $\Delta$, such that $\text{cl}(\text{int}(A)) = \text{cl}(A)$. Then,
\[
\lim_{N \to \infty} \frac{1}{O} \ln \Pr \left( (a_O, c_O) \in A \right) = - \inf_{(a', c') \in A} I(a', c')
\]

We now prove lemma 7.

**Proof.** Let $e_{jk|\tilde{O}}$ and $e_{jk}$ be as defined in section 3, and let $e_{jk|O}$ denote the set obtained by replacing $\tilde{M}$, $\tilde{O}$ and $\tilde{j}_O$ respectively with $M$, $O$ and $j_O$ in $e_{jk|\tilde{O}}$.

Then,
\[
\lim_{N \to \infty} \frac{1}{O} \ln p_{jk}^N = \lim_{N \to \infty} \frac{1}{O} \ln \Pr \left( (a_O, c_O) \in e_{jk|O} \right)
\]

Clearly, $\text{cl}(\text{int}(e_{jk})) = \text{cl}(e_{jk}) = e_{jk}$, since $e_{jk}$ is a compact set, and $\text{int}(e_{jk}) \subseteq \lim_{N \to \infty} e_{jk|O} \subseteq \text{cl}(e_{jk})$, and so, by Sanov’s Theorem,
\[
- \inf_{(a', c') \in \text{int}(e_{jk})} I(a', c') = \lim_{N \to \infty} \frac{1}{O} \ln \Pr \left( (a_O, c_O) \in \text{int}(e_{jk}) \right) \leq \\
\lim_{N \to \infty} \frac{1}{O} \ln \Pr \left( (a_O, c_O) \in e_{jk|O} \right) \leq \\
\lim_{N \to \infty} \frac{1}{O} \ln \Pr \left( (a_O, c_O) \in \text{cl}(e_{jk}) \right) = - \inf_{(a', c') \in \text{cl}(e_{jk})} I(a', c')
\]

The continuity of $I$ implies that
\[
- \inf_{(a', c') \in \text{int}(e_{jk})} I(a', c') = - \inf_{(a', c') \in \text{cl}(\text{int}(e_{jk}))} I(a', c')
\]

Since $\text{cl}(\text{int}(e_{jk})) = \text{cl}(e_{jk}) = e_{jk}$ and $I$ is continuous it follows that
\[
\lim_{N \to \infty} \frac{1}{O} \ln p_{jk}^N = - \min_{(a', c') \in e_{jk}} I(a', c')
\]

**Appendix B**

**Proof of Proposition 6.** Let $j' = \frac{jN}{O}$, $j_N = a_N, b_N, c_N$. Thus, $a' + b' + c' = 1$. Let $I(a', c') = a' \ln \frac{a'}{\pi_A} + (1 - a' - c') \ln \left(1 - \frac{a' - c'}{\pi_B}\right) + c' \ln \frac{c'}{\pi_C}$

be the (unique) rate function of the trinomial distribution, where $b' = 1 - a' - c'$ was used in the expression for $I$, and $0 \ln \frac{0}{\pi_j}$ is conventionally defined to be 0 for all $\pi_j \in [0, 1]$. 

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i) If \( \lim_{N \to \infty} \frac{M}{O} = \frac{ql_{AB}}{ql_{AB}} = x \geq 1 \), then for \( N \) large enough, \( \frac{M}{O} \geq 1 \), which implies that voting for \( A \) yields \( \tilde{a} + \tilde{M} \geq \max \{ b, c \} \).

Thus, for \( N \) large enough, \( p_{AC}^N = p_{AB}^N = 0 \), and an \( AB \)-type will always vote for \( A \).

For the remaining statements, we consider two cases:

**Case 1:** Let \( \pi_j > 0, j \in \{A, B, C\} \).

ii) Let \((\pi_A, \pi_C) \in E_1\).

Let \( ql_{AB} < \frac{1}{3} \). From lemma 7, \( \lim_{N \to \infty} \frac{1}{N} \ln p_{jk}^N = - \min_{(a', c')} I(a', c') \).

Note that \( I(., .) \) is strictly convex and continuously differentiable in \( \text{int} (\Delta) \), which implies that its level curves are closed and smooth.

From strict convexity and differentiability of \( I \) - since these conditions imply smooth closed level curves of \( I \) - the problems

\[
-\min_{(a', c') \in E_{CB}} I(a', c') \quad \text{and} \quad -\min_{(a', c') \in E_{AB}} I(a', c')
\]

both admit unique optimizers in the statistically nearest points to \( I \), and we can solve:

\[
-\min_{(a', c') \in E_{CB}} I(a', c') \Leftrightarrow \begin{cases} 
- \min I(a', c') \\
\text{s.t.} c' = \frac{1}{2} - \frac{c}{2} \\
0 \leq a' \leq 1 - 2c
\end{cases} \quad (11)
\]

\[
-\min_{(a', c') \in E_{AB}} I(a', c') \Leftrightarrow \begin{cases} 
- \min I(a', c') \\
\text{s.t.} c' = 1 - x - 2a' \\
\frac{1 - 2x}{3} \leq a' \leq \frac{1 - x}{2}
\end{cases} \quad (12)
\]

Note that \( p_{AC}^N << p_{CB}^N \) and \( p_{AC}^N << p_{AB}^N \), since all points in \( E_{AC} \) are statistically more distant from \((\pi_A, \pi_C) \in E_1\) than are the optimizers of 11 and of 12.

We now solve both maximization problems for values of \((\pi_A, \pi_C) \in E_1\), for which the unique \( \arg \min_{(a', c') \in E_{jk}} I(a', c') \)

\( I(a', c') \) is not any of the extreme points of the line segments.

Note that the strict convexity of \( I \) and the smoothness of its closed level curves imply that these optimizers are the tangency points between a level curve of \( I \) and the line segments.

Trivially, if \((\pi_A, \pi_C) \in E_{CB} \), then \( \lim_{N \to \infty} \frac{1}{N} \ln p_{CB}^N = 0 \) and an \( AB \)-type votes \( B \), and if \((\pi_A, \pi_C) \in E_{AB} \), then \( \lim_{N \to \infty} \frac{1}{N} \ln p_{AB}^N = 0 \), and an \( AB \)-type votes \( A \).

We now solve the generic problem

\[
\begin{cases} 
\min I(a', c') \\
\text{s.t.} c' = ma' + y
\end{cases} \quad (13)
\]

The Lagrangian is \( L(a', c', \mu) = a' \ln \frac{a}{\pi_A} + (1 - a' - c') \ln \frac{(1 - a' - c')}{\pi_B} + c' \ln \frac{c'}{\pi_C} + \mu (ma' + y - c') \), and the first order conditions (FOC) are

\[
\begin{cases} 
\frac{\partial L}{\partial a'} = \ln a' + 1 - \ln \pi_A - \ln (1 - a' - c') - 1 + \ln \pi_B + m \mu = 0 \\
\frac{\partial L}{\partial c'} = - \ln (1 - a' - c') - 1 + \ln \pi_B + \ln c' + 1 - \ln \pi_C - \mu = 0 \\
\frac{\partial L}{\partial \mu} = ma' + y - c' = 0 \quad (14)
\end{cases}
\]

The solution \((a^*, c^*, \mu^*)\) is given by the following equivalent conditions

\[
\begin{cases} 
\frac{(ma^* + y)^{ma^*}}{(1 - y - (1 + m)a^*^{ma^*})^{ma^*}} = \frac{\pi_A^{a^*}}{\pi_B^{c^*}} \\
c^* = ma^* + y \\
\mu^* = \ln \left( \frac{c^* \pi_B}{(1 - a^* - c^*) \pi_C} \right) \quad (15)
\end{cases}
\]

and

\[
I(a^*, c^*) = \ln \left( \frac{1 - a^* - c^*}{\pi_B} \right)^{1 - y} \left( \frac{c^*}{\pi_C} \right)^y = \ln \left[ \frac{a^*(1 - y)}{\pi_A^{(1 - y)} (1 + m)} \right] \left( \frac{c^*}{\pi_C^{(1 - y)} (1 + m)} \right) \quad (16)
\]

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Replacing \( m = -\frac{1}{2} \) and \( y = \frac{1}{2} \), the solution for 11 is given by

\[
a^* = \frac{\pi_A}{\pi_A + 2\sqrt{\pi_B\pi_C}}
\]

\[
c^* = \frac{\sqrt{\pi_B\pi_C}}{\pi_A + 2\sqrt{\pi_B\pi_C}}
\]

\[
I_{CB}(a^*, c^*) = \ln \frac{1}{\pi_A + 2\sqrt{\pi_B\pi_C}}
\]

Replacing \( m = -2 \) and \( y = 1 - x \), the solution for 12 is given by

\[
a^* = \frac{1}{\pi_C - 4\pi_A\pi_B} \left( \frac{1}{2}\pi_C \sqrt{x^2\pi_C^2 + 4\pi_A\pi_B - 4x^2\pi_A\pi_B} - 2\pi_A\pi_B - \frac{1}{2}x\pi_C^2 + 2x\pi_A\pi_B \right)
\]

\[
c^* = \frac{\pi_C}{\pi_C - 4\pi_A\pi_B} \left( \pi_C - \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right)
\]

\[
I_{AB}(a^*, c^*) = \ln \left( \frac{1}{\pi_C - 4\pi_A\pi_B} \left( \pi_C - \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right) \right) \times
\]

\[
\times \left( \frac{1}{2\pi_B - 2x\pi_B} \left( x\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right) \right).
\]

From 3, an \( AB \)-type will vote for \( A \) as long as \( \lambda_{AB} < \overline{\lambda}_{AB} = \lim_{N \to \infty} \frac{\lambda_{AC}^N}{\lambda_{AB}^N} \).

Now, if \( I_{CB}(a^*, c^*) < I_{AB}(a^*, c^*) \), \( \lim_{N \to \infty} \frac{\lambda_{AC}^N}{\lambda_{AB}^N} = \infty \) and \( \overline{\lambda}_{AB} = 0 \), and if \( I_{CB}(a^*, c^*) > I_{AB}(a^*, c^*) \), \( \lim_{N \to \infty} \frac{\lambda_{AC}^N}{\lambda_{AB}^N} = 0 \) and \( \overline{\lambda}_{AB} = 1 \) - any point in \( e_{AC} \) is statistically more distant than \( \arg \min \ I(a', c'), \) for all \( (\pi_A, \pi_C) \in E_1 \cap \text{int}(\Delta) \). Solving \( I_{CB}(a^*, c^*) = I_{AB}(a^*, c^*) \) yields

\[
\frac{1}{\pi_A + 2\sqrt{\pi_B\pi_C}} = \left( \frac{1}{\pi_C - 4\pi_A\pi_B} \left( \pi_C - \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right) \right) \times
\]

\[
\times \left( \frac{1}{2\pi_B - 2x\pi_B} \left( x\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right) \right),
\]

\[
\Rightarrow \ln \left( \frac{\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B}}{(1 - x^2)(\pi_A + 2\sqrt{\pi_B\pi_C})} \right)
\]

\[
= \frac{1}{2\pi_B - 2x\pi_B} \left( x\pi_C + \sqrt{x^2\pi_C^2 - 4\pi_A\pi_Bx^2 + 4\pi_A\pi_B} \right).
\]

From strict convexity of \( I \), and from the fact that \( I \) has closed and smooth level-curves, \( I_{AB}(a^*, c^*) \) increases in \( 1 - x \), that is, decreases in \( x = \frac{q_{AB}}{1 - q_{AB}} \), which is strictly increasing in \( q \). Therefore, \( I_{AB}(a^*, c^*) \) decreases in \( q \).

Hence, \( \lim_{N \to \infty} \frac{\lambda_{AC}^N}{\lambda_{AB}^N} = \begin{cases} \infty, q < \overline{q} \\ 0, q > \overline{q} \end{cases} \), where \( \overline{q} \) is given by 8, and we obtain the cutoff

\[
\overline{\lambda}_{AB} = \lim_{N \to \infty} \frac{1 + \frac{\lambda_{AC}^N}{\lambda_{AB}^N}}{1} = \begin{cases} 0, q < \overline{q} \\ 1, q > \overline{q} \end{cases}.
\]

Hence, an \( AB \)-type will vote for \( A \) if \( q > \overline{q} \) and for \( B \) if \( q < \overline{q} \).
If $q_{AB} \in \left[\frac{1}{2}, \frac{1}{3}\right]$, $E_1 = \{(\pi_A, \pi_C) \in \Delta : \pi_B \geq x + \pi_A\}$ and in any level curve tangent to a point in $c' = 1 - x - a'$, $I$ is smaller than $I_{CB}(a^*, c^*)$.

Thus, $I_{CB}(a^*, c^*) > I_{AB}(a^*, c^*)$ and an $AB$--type will vote for $A$.

(iii) Let $(\pi_A, \pi_C) \in E_2$.

Let $q_{AB} < \frac{1}{2}$. The proof goes exactly as in (ii), only now $\lim_{N \to \infty} p_{AB}^N = 0$, since any point in $e_{AB}$ is more distant from $(\pi_A, \pi_C) \in E_2$ than $\arg \min (\pi'_A, c')$.

Also, if $(\pi_A, \pi_C) \in E_{CB}$, then $\lim_{N \to \infty} \frac{1}{N} \ln p_{CB}^N = 0$ and an $AB$--type votes $B$, and if $(\pi_A, \pi_C) \in E_{AC}$, then $\lim_{N \to \infty} \frac{1}{N} \ln p_{AC}^N = 0$, and an $AB$--type votes $A$. We now solve

$$- \min_{(a', c') \in e_{CB}} I(a', c') \equiv \begin{cases} - \min I(a', c') & \text{s.t. } c' = \frac{1+x}{2} - \frac{a'}{2} \\ 0 \leq a' \leq \frac{1-x}{3} \end{cases}$$

$$- \min_{(a', c') \in e_{AC}} I(a', c') \equiv \begin{cases} - \min I(a', c') & \text{s.t. } c' = x + a' \\ \frac{1-x}{3} \leq a' \leq \frac{1-x}{2} \end{cases}$$

Replacing $m = -\frac{1}{2}$ and $y = \frac{1+x}{2}$ in 15, the solution for 18 yields

$$a^* = \pi_A - \pi_A - \sqrt{x^2 \pi_A^2 + 4 \pi_B \pi_C - 4x^2 \pi_B \pi_C}$$

$$c^* = \frac{1}{2} \pi_A \sqrt{x^2 \pi_A^2 + 4 \pi_B \pi_C - 4x^2 \pi_B \pi_C - 4 \pi_B \pi_C + x \pi_A^2 - 4 \pi_B \pi_C}$$

$$I_{CB}(a^*, c^*) = \ln \left( \frac{1}{\pi_A - \sqrt{x^2 \pi_A^2 - 4 \pi_B \pi_C x^2 + 4 \pi_B \pi C}} \right) \times$$

$$\times \left( \frac{x \pi_B + \sqrt{x^2 \pi_A^2 + 4 \pi_B \pi_C} - 4 \pi_B \pi_C}{2 \pi_C (1-x)} \right)$$

Replacing $m = 1$ and $y = x$ in 15, the solution for 19 yields

$$a^* = \frac{1}{\pi_B - 4 \pi_A \pi_C} \left( \frac{1}{2} \pi_B \sqrt{x^2 \pi_B^2 + 4 \pi_A \pi_C - 4x^2 \pi_A \pi C - 2 \pi_A \pi_C} - \pi_B \sqrt{x^2 \pi_B^2 + 2 \pi_B \pi_A \pi C} \right)$$

$$c^* = \frac{1}{2} \pi_B \sqrt{x^2 \pi_B^2 + 4 \pi_A \pi_C - 4x^2 \pi_A \pi C - 4 \pi_A \pi C + x \pi_B^2 - 4 \pi_A \pi C}$$

$$I_{AC}(a^*, c^*) = \ln \left( \frac{1}{\pi_B - \sqrt{x^2 \pi_B^2 - 4 \pi_A \pi C x^2 + 4 \pi_A \pi C}} \right) \times$$

$$\times \left( \frac{x \pi_B + \sqrt{x^2 \pi_B^2 - 4 \pi_A \pi C x^2 + 4 \pi_A \pi C}}{2 \pi_C (1-x)} \right)$$

Due to the strict convexity of $I$, and to the smoothness of its closed level curves, $I_{CB}(a^*, c^*) = I_{AC}(a^*, c^*)$ has a unique solution. Since $I_{CB}(a^*, c^*)$ and $I_{AC}(a^*, c^*)$ are symmetric in the exchange of $\pi_B$ and $\pi_A$, $I_{CB}(a^*, c^*) = I_{AC}(a^*, c^*) \Rightarrow \pi_B = \pi_A$.\textsuperscript{19} Hence,
\[ \overline{\lambda}_{AB} = \lim_{N\to\infty} \frac{p_{AB}^N}{p_{CB}^N + p_{AB}^N} = \begin{cases} \infty, & \pi_A > \pi_B \\ 0, & \pi_A < \pi_B \end{cases} \]

and an \( AB \)-type votes \( A \) if \( \pi_A > \pi_B \) and votes \( B \) if \( \pi_A < \pi_B \).

\( iv \) Let \((\pi_A, \pi_C) \in E_3.20\) Then,

\[ \overline{\lambda}_{AB} = \lim_{N\to\infty} \frac{1 + p_{AB}^N}{p_{AB}^N} \geq 1, \text{ since } \lim_{N\to\infty} p_{AB}^N = 0 \]

(Note that \( \min_{(a',c') \in CB} I(a',c') > \min_{(a',c') \in AB} I(a',c'). \))

Case 2: In order to complete the proof, we must establish the RBR in case any \( \pi_j \) is zero.

\( ii \) Let \( \pi_A = 0 \). Then, \( I(c') = (1-c') \ln \frac{(1-c')}{\pi_B} + c' \ln \frac{c'}{\pi_C} \) is the rate function of the binomial distribution.

We have that \( E_1 \) simplifies to \( E_1 = \{ \pi_C : \pi_C \leq \frac{1}{2} \wedge \pi_C \leq 1-x \wedge x \leq 1 \} \) and problems 11 and 12 respectively reduce to

\[ \{ -\min I(c') \} \]

\[ \text{s.t. } c' = \frac{1}{2} \]

and

\[ \{ -\min I(c') \} \]

\[ \text{s.t. } c' = 1-x . \]

If \( 1-x > \frac{1}{2} \leftrightarrow x < \frac{1}{2} \), then \( I(\frac{1}{2}) < I(1-x) \) and an \( AB \)-type votes for \( B \). Conversely, if \( x > \frac{1}{2} \), an \( AB \)-type votes for \( A \).

Let \( \pi_C = 0 \). An \( AB \)-type votes for \( A \), since \( C \) gets zero votes.

\( iii \) Let \( \pi_A = 0 \). Then \( E_2 = \{ \pi_C : \pi_C \geq \frac{1-x}{2} \wedge \pi_C \geq x \wedge x < 1 \} \) and problems 18 and 19 respectively reduce to

\[ \{ -\min I(c') \} \]

\[ \text{s.t. } c' = \frac{1-x}{2} \]

and

\[ \{ -\min I(c') \} \]

\[ \text{s.t. } c' = x . \]

If \( \frac{1-x}{2} > x \leftrightarrow x < 1 \), then \( I(\frac{1-x}{2}) < I(x) \) and an \( AB \)-type votes for \( B \). As established in \( i \), if \( x > 1 \), an \( AB \)-type votes for \( A \).

Let \( \pi_B = 0 \). Then an \( AB \)-type votes \( A \): since \( \bar{a} \geq \bar{b} = 0 \), if by voting for \( A \) an \( AB \)-type cannot ensure that \( A \) beats \( C \), then \( B \) will also lose to \( C \) with her votes, and voting for \( B \) would be a dominated action.

\( iv \) If \( \pi_A = \pi_C = 0 \Rightarrow \bar{c} = 0, \bar{p}_{CB}^N = 0 \) and from lemma 4 an \( AB \)-type votes for \( A \).

If \( \pi_A = 0 \) and \( \pi_C = 1 \Rightarrow \bar{c} = \bar{O}, \bar{b} = 0, \bar{p}_{CB}^N = 0 \), since \( \bar{b} = 0 \geq \bar{c} - \bar{M} > 0 \) is impossible, and from lemma 4 an \( AB \)-type votes \( A \).

This completes the proof. ■

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\[ ^{20}\text{Note that } \pi_A > 0. \]


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