Nash and Limit Equilibria of Games with a Continuum of Players*

Guilherme Carmona

Universidade Nova de Lisboa

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Abstract

We show that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an $\varepsilon_n$-equilibrium, with $\varepsilon_n$ converging to zero. In our characterization, the sequence of finite games approaches the continuum game in the sense that the set of players and the distribution of characteristic and actions in the finite games converge to those of the continuum game. These results render approximate equilibria of large finite economies as an alternative way of obtaining strategic insignificance. Also, they suggest defining a refinement of Nash equilibria for games with a continuum of agents as limit points of equilibria of finite games. This allows us to discard those Nash equilibria that are artifacts of the continuum model, making limit equilibrium a natural equilibrium concept for games with a continuum of players.

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1 Introduction

Many economic situations evolve a large number of participants, each of which having a negligible influence on the aggregate outcome. As Aumann [1] has convincingly argued, the ideal situation of strategic insignificance can only be obtained in models featuring a continuum of agents. Likewise, equilibrium concepts that depend on the idea of strategic insignificance make good sense only in those models. This makes models with a continuum of agents an appealing framework for economic analysis.

Of course, real economies have finitely many agents. Hence, conclusions obtained by studying economies with a continuum of agents will, typically, hold only approximately on real, finite economies. It is, thus, of central importance to know to what extent are continuum-of-agents economies good approximations of large finite economies. For equilibrium analysis, in particular, it is important to know whether the action profiles that a given equilibrium concept singles out as reasonable outcomes in a continuum of agents economy are also reasonable in some sense in large finite economies.\(^1\)

Moreover, models with finitely many agents are more intuitive and therefore easier to understand than models with a continuum of agents. The same is true regarding equilibria of those models. Thus, we ask: can we relate equilibria of models with a continuum of agents with the more intuitive notion of equilibria of finite models?

We provide an answer to the above question for normal form games in which the payoff of each player depends on his choice and on the distribution of actions (see Mas-Colell [21]). First, we characterize Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with finitely many players. Our result shows, in particular, that any Nash equilibrium of the continuum of players game is an approximate equilibrium in some sequence of similar finite games. Second, we provide an alternative characterization for games with a continuum of players in which there a bound on the diversity of payoffs;\(^2\) in particular, this second characterization shows that to any Nash equilibrium of the continuum-of-players game we can associate another, equivalent Nash equilibrium which is an approximate equilibrium in all sequences of similar finite games. Third, we show (under stronger assumptions) that there exist Nash equilibria that are

\(^1\)This is just one possible approximation question. In fact, others have been considered in the literature, and will be revised in section 2.

\(^2\)Formally, players’ payoff functions form an equicontinuous family.
limit points (in a particular sense described below) of Nash equilibria of finite games.

The two characterization results have the potential to make games with a continuum of players accessible to researchers that are not familiar with measure theoretical tools needed to analyze such games. This is specially the case when there are finitely many different payoff functions and possible actions — as we show later on, in that case the only tools needed are the usual notions of convergence in the real line and approximate equilibrium in a finite normal form game.

Furthermore, our results suggest two ways of expressing the idea that conclusions obtained in games with a continuum of players hold approximately in some close finite games: first, conclusions reached using Nash equilibrium hold in approximate equilibria; second, those reached by limit equilibria also hold approximately in similar strategies that are Nash equilibria of finite games.

Despite the above interpretation of our characterization results, we point out that they don’t imply that all Nash equilibria of games with a continuum of players will have similar properties to those of large finite games. In fact, we present two simple examples in which some Nash equilibria of games with a continuum of players have very different properties compared to those of Nash equilibria of similar large finite games. This difficulty with Nash equilibrium can, in principle, be corrected by focusing on limit equilibria, which appears to be a better notion of equilibrium for games with a continuum of players.

A more precise description of our results, which we give in what follows, will allow us to make some additional remarks. Our first result shows that a strategy is a Nash equilibrium in a game with a continuum of players if and only if there exists a sequence of finite games such that its restriction is an $\varepsilon_n$-equilibrium, with $\varepsilon_n$ converging to zero. Thus, Nash equilibria of games with a continuum of agents are exactly the strategies that are approximate equilibria in some games obtained from the original one by selecting finitely many agents. In particular, this result give us a sense in which approximate equilibria of large finite games is an alternative way of obtaining strategic insignificance. This interpretation is strengthened in equicontinuous games: in those games, every Nash equilibrium can be changed in a set of measure zero to obtain another Nash equilibrium for which the above characterization holds for all approximating sequences of finite games.

The sequence of finite games in the above results approximates the strate-
gic situation described by the given strategy in the game with a continuum of agents in the following sense: the set of players and distribution of characteristic and actions in the finite games converge to those of the continuum game.\textsuperscript{3} The notion of approximation of games is important because, although there are many finite economies which we can associate with a continuum game, only those finite games that can describe approximately the same economic situation as the continuum game should be regarded as reasonable approximations. Since we obtain a compete characterization of equilibria, we can indeed interpret this approximation as convergence of the strategic situation in the finite games to the one in the continuum game.

This convergence also allow us to define a refinement of Nash equilibrium for games with a continuum players in the same spirit of Selten’s \textsuperscript{[31]}’s perfect equilibrium: we say that a strategy is a limit equilibrium if it is the limit point in the above sense of a sequence of equilibria in finite games. Intuitively, limit equilibria are those that inherit the properties of equilibria of finite games, and so, by using it, we can in principle discard those Nash equilibria that are an artifact of the continuum construction. This point is illustrated by two examples, all of which having a continuum of Nash equilibria but a single, and plausible, limit equilibrium.

The interest of limit equilibrium would, nevertheless, be limited without an existence result. We show that in games in which the action space is finite, any strict equilibrium\textsuperscript{4} is a limit equilibrium, thus providing a sufficient conditions that guarantee the existence of limit equilibria. We also show that a limit equilibrium distribution exists in quasi-concave games; finally, we show that a limit equilibrium distribution can be represented as a limit equilibrium whenever it has a countable support.

In summary, we show that the conclusions obtained by studying Nash equilibria of games with a continuum of players hold in approximate equilibria in some similar large finite games. Despite this result, simple examples show that at least some Nash equilibria of games with a continuum of players can lead to very different conclusions than those obtained in large finite games. As the construction of limit equilibrium is designed to overcome this problems, it makes it a more natural notion of equilibrium for games with a

\textsuperscript{3}Instead of requiring that the set of players in the finite games converges to that of the continuum game, we can require only that the number of players in the finite games converges to infinity.

\textsuperscript{4}I.e., a Nash equilibrium in which almost all players strictly prefer their action to any other.
continuum of players.

2 Related Literature

Our results establish a relationship between equilibria of a continuum game with approximate equilibria of finite games. They are thus related with the paper of Fudenberg and Levine [9] which characterize equilibria of a game as the limit points of games with smaller strategy space. As typically these smaller strategy spaces are finite whereas they are infinite in the game of interest, their results also establish a relationship between equilibria of a continuum game with approximate equilibria of finite games.

A related question is whether the limit of a converging sequence of equilibria of finite economies is an equilibrium in the continuum economy. This question was studied by Hildenbrand and Mertens [18] for pure exchange economies, by Dubey et al. [8] for strategic market games, and by Green [11] for normal form games. As those authors pointed out, this question amounts to asking whether the equilibrium correspondence is upper hemicontinuous when viewed as a (set-valued) function on the set of measures on the set of economic agents; general theorems on the upper hemicontinuity of the equilibrium correspondence were established in those papers.

Our work is also related with whether the equilibrium correspondence is lower hemicontinuous. In general, the equilibrium correspondence will not have that property. However, the papers by Mas-Colell [20] and Novshek and Sonnenschein [23] show that regular Walrasian equilibria of a continuum economy can be approximated by the noncooperative Cournot equilibria for the tail of any approximating sequence of finite economies, a property that is close to lower hemicontinuity.

Postlewaite and Schmeidler [25] and Hildenbrand [17] showed that large finite games have properties that are approximate versions of those of continuum economies. For example, Postlewaite and Schmeidler [25] showed that Nash equilibria of large finite market games are approximately efficient, a property that equilibria of market games with a continuum of player have. Also, Rashid [26] and Wooders, Selten and Cartwright [33] showed that large finite games in a certain class have Nash equilibria in which all but a small fraction of players play pure strategies; this result clearly parallels Schmeidler [30]’s Theorem 2, asserting that any game with a continuum of players of the same class have a Nash equilibrium in which almost all players play a
In Barlo and Carmona [3] we propose a refinement of Nash equilibria which is similar to limit equilibria. There we consider perturbed games that have a continuum of players, and that differ from the original one because every player believes that he alone has a small, but positive, impact on the societal choice. We define a strategic equilibrium as a limit point of Nash equilibria of these perturbed games when players’ impact on the societal choice goes to zero. Although there seems to be a relation between the two approaches, its precise nature is still unknown to us.

In Carmona [4], we consider a more specialized framework, in which each player’s payoff functions depend only on his action and on the average choice of the others. There we obtain similar characterization results for different notions of approximation of games, which can be thought of as alternative ways of describing the convergence of the economic situation. In Carmona [5] we use similar tools to the ones used here to characterize Nash equilibrium distributions of games with a continuum of players in terms of symmetric, approximate equilibrium distributions with finite support of similar continuum-of-players games.

On the technical side, we have gain much form the suggestion by Hildenbrand [17], Hart, Hildenbrand and Kohlberg [15] and Mas-Colell [21] that any game with a continuum of agents are easily analyzed by studying the distributions it induces on the space of action and on the space of players’s characteristics. Such analytical tool is helpful because the distribution induced by a game and an equilibrium strategy is an equilibrium distribution, and conversely, if an equilibrium distribution is the distribution induced by a game and a strategy, then this strategy is an equilibrium of that game. Also, as in Hildenbrand [16] and as in Debreu [7], we use the weak convergence of measures and the convergence of closed set with respect to the Hausdorff distance in order to define our notion of convergence of games.

3 Large Games

Let $A$ be a non-empty, compact metric space of actions and $\mathcal{M}$ be the set of Borel probability measures on $A$ endowed with the weak convergence topol-

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5Rashid [26] and Wooders, Selten and Cartwright [33] showed a stronger result: all Nash equilibria of sufficiently large finite games can be purified in $\varepsilon$-equilibria. See also Carmona [6].
ogy. By Parthasarathy [24, Theorem II.6.4], it follows that $\mathcal{M}$ is a compact metric space. We use the following notation: we write $\mu_n \Rightarrow \mu$ whenever $\{\mu_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ converges to $\mu$ and $\rho$ denote the Prohorov metric on $\mathcal{M}$, which is known to metricize the weak convergence topology. We let $d_A$ denote the metric on $A$.

Let $\mathcal{U}$ denote the space of continuous utility functions $u : A \times \mathcal{M} \to \mathbb{R}$ endowed with the supremum norm. The set $\mathcal{U}$ represents the space of players characteristics; it is a complete, separable metric space.

A game with a continuum of players is characterized by a measurable function $U : [0, 1] \to \mathcal{U}$, where the unit interval $[0, 1]$ is endowed with the Lebesgue measure $\lambda$ on the Lebesgue measurable sets and represents the set of players. We represent such game by $G = (([0, 1], \lambda), U, A)$.

A game with a finitely many players is characterized by a function $U : T \to \mathcal{U}$, where $T$ is a finite subset of $[0, 1]$. The set $T$ represents the set of players and it is endowed with the uniform measure $\nu$: if $T$ has $N$ elements, then the measure $\nu$ on $T$ satisfies $\nu(\{t\}) = 1/N$ for all $t \in T$. We represent such game by $G = ((T, \nu), U, A)$.

We are especially interested in games with finitely many players that are derived from a given game $G = (([0, 1], \lambda), U, A)$ with a continuum of players. Those are games $H = ((T, \nu), U|_T, A)$ where $U|_T$ denotes the restriction of $U$ to $T$.

In all the cases above, a game is defined as a measurable function from a measure space into $\mathcal{U}$. Although we will focus exclusively in the particular cases mentioned above, we present the following definition in this general case.

Let $(X, \mathcal{X}, \mu)$ be a measure space and $G = ((X, \mathcal{X}, \mu), U, A)$ be a game. A strategy is a measurable function $f : X \to A$. Given a strategy $f$, $y \in A$, and $t \in T$, let $f \setminus_t y$ denote the strategy $g$ defined by $g(t) = y$, and $g(\tilde{t}) = f(\tilde{t})$, for all $\tilde{t} \neq t$.

For any $\varepsilon \geq 0$ and strategy $f$ let

$$E(f, \varepsilon, \mu) = \left\{ t \in \text{supp}(\mu) : U(t)(f(t), \mu \circ f^{-1}) \geq U(t)(a, \mu \circ (f \setminus_t a)^{-1}) - \varepsilon \text{ for all } a \in A \right\}.$$  \hspace{1cm} (1)

The set $E(f, \varepsilon, \mu)$ is the set of players in the support of $\mu$ that are within $\varepsilon$ of their best response by playing according to $f$. When $\varepsilon = 0$, we will write $E(f, \mu)$ instead of $E(f, 0, \mu)$.
Lemma 1 Let $\varepsilon \geq 0$, a game $G = ([0,1], \lambda, U, A)$, and a strategy $f$ be given. Then, $E(f, \varepsilon, \lambda)$ is measurable.

It is clear that $E(f, \varepsilon, \mu)$ is measurable when $\mu$ has finite support. Hence, $\mu(E(f, \varepsilon, \mu))$ is well-defined both when $\mu = \lambda$, and when $\mu$ has finite support. We then say that $f$ is an $\varepsilon$--equilibrium of a game $G$ if $\mu(E(f, \varepsilon, \mu)) \geq 1 - \varepsilon$. Thus, in an $\varepsilon$--equilibrium, all but a small fraction of players are close to their optimum by choosing according to $f$. A strategy $f$ is a Nash equilibrium of $G$ if $f$ is an $\varepsilon$--equilibrium of $G$ for $\varepsilon = 0$.

4 Equilibrium Distributions

4.1 Games with a continuum of players

Instead of defining a game as a measurable function from players into characteristic, we could have started by describing the game as a probability measure $\mu$ on $U$ as in Mas-Colell [20]. For our purpose, equilibrium distributions provide a useful device to study properties of equilibria in games with a continuum of agents.

Given a Borel probability measure $\tau$ on $U \times A$, we denote by $\tau_U$ and $\tau_A$ the marginals of $\tau$ on $U$ and $A$ respectively. The expression $u(a, \tau) \geq u(A, \tau)$ means $u(a, \tau) \geq u(a', \tau)$ for all $a' \in A$.

Given a game $\mu$, a Borel probability measure $\tau$ on $U \times A$ is an equilibrium distribution for $\mu$ if

1. $\tau_U = \mu$, and
2. $\tau(\{ (u, a) \in U \times A : u(a, \tau_A) \geq u(A, \tau_A) \}) = 1$.

We will use the following notation: $B_\tau = \text{supp}(\tau) \cap \{ (u, a) \in U \times A : u(a, \tau_A) \geq u(A, \tau_A) \}$. Note that $B_\tau$ is closed, and so a Borel set; hence $\tau(B_\tau)$ is well defined. Also, if $(u, a)$ belong to $B_\tau$, then $a$ maximizes the function $\tilde{a} \mapsto u(\tilde{a}, \tau_A)$. Thus, we are implicitly assuming that the choice of any player does not affect the distribution of actions. It is in this sense that the notions of this section describe a game with a continuum of players.

Any game $G = ([0,1], \lambda, U, A)$ and strategy $f$ induces a Borel probability measure $\tau$ on $U \times A$ by the formula $\tau = \lambda \circ (U, f)^{-1}$. Furthermore, as the next lemma show, if $f$ is a Nash equilibrium of $G$ then $\tau = \lambda \circ (U, f)^{-1}$.
is an equilibrium distribution of \( \lambda \circ U^{-1} \); conversely, if \( \tau \) is an equilibrium distribution and \( \tau = \lambda \circ (U, f)^{-1} \) then \( f \) is a Nash equilibrium of \( G \).

**Lemma 2** A strategy \( f \) is a Nash equilibrium of a game \( G = ((([0, 1], \lambda), U, A) \) if and only if \( \tau = \lambda \circ (U, f)^{-1} \) is an equilibrium distribution of \( \lambda \circ U^{-1} \).

### 4.2 Games with finitely many players

Similarly as for games with a continuum of players, any games with finitely many players together with a strategy also induces a Borel probability measure on \( U \times A \), again by the formula \( \tau = \nu \circ (U, f)^{-1} \). However, in such game, the choice of a single player has an affect on the distribution of actions and the definition of an equilibrium distributions needs to be adapted accordingly.

Let \( G = ((T, \nu), U, A) \) be a game with finitely many players, \( f \) a strategy, \( \tau = \nu \circ (U, f)^{-1} \) and \( \varepsilon > 0 \). Then \( \tau \) is an \( \varepsilon \)-equilibrium distribution of \( \nu \circ U^{-1} \) if \( \tau(\{(u, a) \in U \times A : u(a, \tau) \geq u(a', \tau_{A}^{u,a,a'}) - \varepsilon \text{ for all } a' \in A\}) \geq 1 - \varepsilon \), where \( \tau_{A}^{u,a,a'} = \nu \circ g^{-1} \), \( g \) is defined by \( g(t') = a' \), and \( g(t) = f(t) \) for all \( t \neq t' \) and \( t' \in T \) is such that \( (U(t), f(t)) = (u, a) \).

Note first that the distribution \( \tau_{A}^{u,a,a'} \) is independent of the choice of \( t' \). This is the distribution on the action space \( A \) that will arise if one player with characteristic \( u \) and playing \( a \) deviates and plays \( a' \). In fact, we can simply define \( \tau_{A}^{u,a,a'} \) as the marginal on \( A \) of \( \tau_{A}^{u,a,a'} \), which is defined from \( \tau \) as follows: \( \tau^{u,a,a'}(\{(u, a)\}) = \tau(\{(u, a)\}) - 1/|T| \), \( \tau^{u,a,a'}(\{(u, a')\}) = \tau(\{(u, a')\}) + 1/|T| \) and \( \tau^{u,a,a'}(\{(\tilde{u}, \tilde{a})\}) = \tau(\{(\tilde{u}, \tilde{a})\}) \) for all \( (\tilde{u}, \tilde{a}) \) different from \( (u, a) \) and from \( (u, a') \). Note that this definition allow us to define an equilibrium distribution of a finite game without the explicit knowledge of a strategy, which will be useful in section 6.2.

The following lemma show that for large finite games \( \tau_{A}^{u,a,a'} \) is close to \( \tau_{A} \).

**Lemma 3** Let \( G = ((T, \nu), U, A) \) be a game with finitely many players and \( f \) a strategy. If \( g \) is another strategy that differs from \( f \) in at most one point, then

\[
\rho(\nu \circ f^{-1}, \nu \circ g^{-1}) \leq \frac{1}{|T|}.
\]

Note also that if the game \( G \) had a continuum of players then \( \tau_{A}^{u,a,a'} = \tau_{A} \) for all \( u, a, a' \) and so in fact this definition coincides with the one given before. Similarly as that case we have:
Lemma 4 A strategy \( f \) is an \( \varepsilon \)-equilibrium of a game \( G = ((T, \nu), U, A) \) if and only if \( \tau = \nu \circ (U, f)^{-1} \) is an \( \varepsilon \)-equilibrium distribution of \( \nu \circ U^{-1} \).

We will use the following notation: \( B^\varepsilon_\tau = \text{supp}(\tau) \cap \{(u, a) \in U \times A : u(a, \tau_A) \geq u(a', \tau^{u,a,a'}_A) - \varepsilon \text{ for all } a' \in A \} \), where \( \tau^{u,a,a'}_A \) is as before.

5 Characterizations of Nash Equilibria

Our first main result is a characterization result. It characterizes Nash equilibria of games with a continuum of players in terms of approximate equilibria of games with finitely many player. As the finite games in the characterization converge (in the sense of properties 2 and 3 below) to the game with a continuum of players, Theorem 1 is also a limit result. Before we state it, we need the following notion of convergence of closed sets, which will be used in our notion of convergence of games: If \( C \) is a non-empty closed set and \( \{C_n\} \) is a sequence of non-empty closed sets then \( C_n \to C \) means convergence with respect to the Hausdorff distance (see Hildenbrand [17, p. 16]).

Theorem 1 A strategy \( f \) is a Nash equilibrium of a game \( G = (((0, 1], \lambda), U, A) \) with a continuum of players if and only if there exists a sequence \( \{\nu_n\}_{n=1}^{\infty} \) of measures, and a sequence \( \{\varepsilon_n\}_{n=1}^{\infty} \) of positive real numbers such that:

1. \( \nu_n \) is the uniform measure on \( T_n \), a finite subset of \([0, 1]\),
2. \( \nu_n \circ (U, f)^{-1} \Rightarrow \lambda \circ (U, f)^{-1} \),
3. \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \)
4. \( \varepsilon_n \downarrow 0 \), and
5. \( f|_{T_n} \) is an \( \varepsilon_n \)-equilibrium of \( G_n = ((T_n, \nu_n), U|_{T_n}, A) \) for all \( n \in \mathbb{N} \).

Theorem 1 provides a natural interpretation of Nash equilibria of games with a continuum of players: if \( f \) is such a strategy, then we can find a finite game, similar to the original continuum one, in which \( f \) is close of being a Nash equilibrium. Conversely, if a strategy \( f \) can be made as closer of being a Nash equilibrium as we want in some finite game similar to the original continuum one, then \( f \) will be a Nash equilibrium of the continuum game. Given this equivalence, it is quite natural that approximate equilibria
of large finite games have approximate the same properties of Nash equilibria of games with a continuum of players, as has been shown by many authors.

This result also confirms Fudenberg and Levine [9]'s conclusions on the appropriate definition of equilibrium in games that are defined as limits. As in their paper, Theorem 1 shows that in order to describe all equilibria of a game with a continuum of players it is necessary to take limits, not only of equilibria of converging finite games, but of $\varepsilon-$equilibria with $\varepsilon$ converging to zero.

**Remark 1** Theorems 1 holds if we replace the condition $\text{supp}(\nu_n) \to \text{supp}(\lambda)$ by the weaker condition that $|T_n| \to \infty$.

Since $|T_n| \to \infty$ is weaker than $\text{supp}(\nu_n) \to \text{supp}(\lambda)$, the remark is clear regarding the necessary condition. Furthermore, it can easily verified that the proof of the sufficiency part only requires $|T_n| \to \infty$.

An important implication of our two characterization results, especially in the format of the above remark, is that they allow us to determine whether a given strategy is a Nash equilibrium of a game with a continuum of agents without necessarily having to deal with the technical difficulties involved in such games. Consider, for instance, a game with a continuum of agents in which there is a finite number of actions and a finite number of possible payoff functions, a typical assumption in applications. In this case, all the tools we need to analyze such game are standard: we need to determine what is the minimal $\varepsilon$ that make a given strategy an $\varepsilon-$equilibrium in a finite normal form game and we need to guarantee that $\nu_n \circ (U|_{T_n}, A)^{-1}(\{(u,a)\})$ converges to $\lambda \circ (U, A)^{-1}(\{(u,a)\})$ (in $\mathbb{R}$) for all pairs $(u,a)$ in $U(\{0,1\}) \times A$.

We illustrate the above comment with the following simple example. Let $G = (((\{0,1\}, \lambda), U, A)$ be described by: $A = \{a, b\}$ and $U(t) = u$ for all $t \in [0,1]$, where $u(a,\mu) = \mu(\{a\})$ and $u(b, \tau) = 1 - \mu(\{a\})$. It is clear that the unique equilibrium distribution of this game is $\tau$ satisfying $\tau(\{(u,a)\}) = \tau(\{(u,b)\}) = 1/2$. Thus, a strategy $f$ defined by

$$f(t) = \begin{cases} a & \text{if } 0 \leq t \leq \frac{1}{2}, \\ b & \text{otherwise} \end{cases}$$

(2)

is a Nash equilibrium. This fact can be inferred by Theorem 1 as follows: for each $n \in \mathbb{N}$, let $t_1^n = 1/2 - 1/(2n)$, $t_2^n = 1/2 + 1/(2n)$, $T_n^1 = \{t_1^n, \ldots, t_1^n\}$, $T_n^2 = \{t_2^n, \ldots, t_2^n\}$ and $T_n = T_n^1 \cup T_n^2$. Letting $\tau_n = \nu_n \circ (U|_{T_n}, f|_{T_n})^{-1}$, we have that $\tau_n(\{(u,a)\}) = \tau_n(\{(u,b)\}) = 1/2$ and so, obviously, $\tau_n \Rightarrow \tau$. For $t \in T_n^1$,
we have $u(f(t), \tau_{A,n}) = 1/2$ if player $t$ plays $f(t)$; if she chooses $b$, then she changes the distribution of actions to $\tau_{A,n}^{a,b}(\{a\}) = 1/2 - 1/(2n)$, thus receiving $u(b, \tau_{A,n}^{a,b}) = 1/2 - 1/(2n)$. Defining $\varepsilon_n = 1/2^n$, we conclude that player $t$ is $\varepsilon_n$-optimizing. Since a similar result holds for any $t \in T_{n}$, it follows that $f|_{T_{n}}$ is an $\varepsilon_n$-equilibrium of $G_{n} = (\{T_{n}, \nu_{n}\}, U|_{T_{n}}, A)$ for all $n \in \mathbb{N}$. Finally, since $\varepsilon_n \searrow 0$, $\tau_{n} \Rightarrow \tau$ and $|T_{n}| \to \infty$, then $f$ is an equilibrium of $G$.

We showed in Theorem 1 that for any Nash equilibrium $f$ we can find a sequence of finite games such that $f$ is an approximate equilibrium in those games. The following question arises naturally: if we are given a Nash equilibrium $f$ and arbitrary sequence of finite games converging in the sense of 1–3 of Theorem 1, when is it the case that $f$ is a $\varepsilon_n$-equilibrium with $\varepsilon_n \to 0$? There are essentially two difficulties with this question: first, players’ characteristics may be too diverse; second, some players that are not optimizing in the limit game by playing according to $f$ may be players in all finite games. We can solve the first problem by adding an equicontinuity assumption; we solve the second by replacing $f$ by an equivalent strategy.

Let $K$ be a subset of $U$. Then $K$ is equicontinuous if if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|u(a, \tau) - u(b, \mu)| < \varepsilon$$

whenever $\max\{d_A(a, b), \rho(\tau, \mu)\} < \delta$, $a, b \in A$, $\tau, \mu \in \mathcal{M}$ and $u \in K$ (see Rudin [29, p. 156]). In our framework, equicontinuity can be interpreted as placing “a bound on the diversity of payoffs,” as pointed out by Khan, Rath and Sun [12].

Let $f$ and $g$ be strategies in a game $G = (([0, 1], \lambda), U, A)$. We say that $g$ is equivalent to $f$, and write $f \sim g$, if $\lambda(\{t \in [0, 1] : g(t) \neq f(t)\}) = 0$ (see Kolmogorov and Fomin [14, 28.3, p. 288]). Note that if $f$ is a Nash equilibrium of $G$ and $f \sim g$, then $g$ is a Nash equilibrium of $G$ as well.

**Theorem 2** Assume that $U([0,1])$ is equicontinuous. Then, a strategy $f$ is a Nash equilibrium of a game $G = (([0,1], \lambda), U, A)$ with a continuum of players if and only if there exists a strategy $g \sim f$ for which the following property holds:

For any sequence $\{\nu_n\}_{n=1}^{\infty}$ of measures satisfying

1. $\nu_n$ is the uniform measure on $T_n$, a finite subset of $[0,1]$,
2. $\nu_n \circ (U|_{T_n}, g|_{T_n})^{-1} = \lambda \circ (U, g)^{-1}$,
3. \(\text{supp}(\nu_n) \to \text{supp}(\lambda)\)

there exists a sequence \(\{\varepsilon_n\}_{n=1}^{\infty}\) of positive real numbers such that \(\varepsilon_n \to 0\), and \(g|_{\mathcal{T}_n}\) is an \(\varepsilon_n\)-equilibrium of \(G_n = ((T_n, \nu_n), U|_{\mathcal{T}_n}, A)\) for all \(n \in \mathbb{N}\).

Again, we remark that Theorem 2 hold if we replace the condition \(\text{supp}(\nu_n) \to \text{supp}(\lambda)\) by the weaker condition that \(|\mathcal{T}_n| \to \infty\). Together with the following remark, it makes Theorem 2 an useful tool to analyze games with a continuum of agents with a finite number of actions and a finite number of possible payoff functions.

**Remark 2** If both \(U([0,1])\) and \(A\) are finite (or more generally, if \(\text{supp}(\lambda \circ (U,f)^{-1})\) is finite), then we can let \(g = f\) in the statement of Theorem 2.

We illustrate how Theorem 2 together with Remark 2 can be used to show that a strategy is not a Nash equilibrium. In the example after Theorem 1, let strategy \(g\) be defined by

\[
g(t) = \begin{cases} 
a & \text{if } 0 \leq t \leq \frac{1}{4}, \\
b & \text{otherwise} \end{cases}
\]

One easily sees that \(g\) is a not a Nash equilibrium. This fact can also be inferred by Theorem 2 as follows: for each \(n \in \mathbb{N}\), let \(t_n^1 = 1/4 - 1/(4n)\), \(t_n^2 = 1/4 + 1/(4n)\), \(T_n^1 = \{t_n^1, \ldots, t_n^1\}\), \(T_n^2 = \{t_n^2, \ldots, t_n^2\}\) and \(T_n = T_n^1 \cup T_n^2\). Letting \(\mu_n = \nu_n \circ (U|_{\mathcal{T}_n}, f|_{\mathcal{T}_n})^{-1}\), we have that \(\mu_n(\{(u,a)\}) = 1/4\) and \(\mu_n(\{(u,b)\}) = 3/4\); obviously, \(\mu_n \Rightarrow \mu\), where \(\mu = \lambda \circ (U,g)^{-1}\). For \(t \in T_n^1\), we have \(u(f(t), \mu_{A,n}) = 1/4\) if player \(t\) plays \(f(t)\); if he chooses \(b\), then he changes the distribution of actions to \(\mu_{A,b}^{a,b}(\{a\}) = 1/4 - 1/(4n)\), thus receiving \(u(b, \mu_{A,n}^{a,b}) = 3/4 + 1/(4n)\). Since \(\nu_n(\{t \in T_n : u(f(t), \mu_{A,n}) < u(a', \mu_{A,n}^{a,a'}) - 1/2 \text{ for } a' \neq a\} \geq 1/4\) it follows that \(g|_{\mathcal{T}_n}\) is not an \(1/4\)-equilibrium of \(G_n\) for all \(n \in \mathbb{N}\). Hence, \(g\) is not a Nash equilibrium of \(G\).

Theorem 2 strengthens the idea that that games with a continuum of players can be useful in order to infer properties about large finite games. In fact, any property that the Nash equilibrium \(g\) has, will hold in approximate equilibrium in all close finite games.

The following theorem build upon this view and presents an asymptotic result on the existence of pure strategy Nash equilibria. The problem can be stated as follows: consider a finite game \(G = ((T,\nu), A, U)\), where \(A\) is a set of mixed strategies. That is, letting \(\Delta_m\) denote the unit simplex in
\( \mathbb{R}^m, m \in \mathbb{N}, \) we have that \( A = \Delta_m \) with the usual interpretation: there are \( m \) pure strategies which are identified with the vertices \( \{e_1, \ldots, e_m\} \) of \( \Delta_m \), and players can randomize over it. Also, we assume the following expected utility hypothesis: if \( a = \sum_{i=1}^{m} a_i e_i \), then \( U(t)(a, \mu) = \sum_{i=1}^{m} a_i U(t)(e_i, \mu) \) for all \( t \in T \) and \( \mu \in \mathcal{M} \).

To any such game \( G \), with \( T = \{t_1, \ldots, t_{|T|}\} \), we associate to it its \( n^{th} \)-replica, \( n \in \mathbb{N} \): we say that a game \( G_n \) is a \( n^{th} \)-replica of \( G \) if the set of players has \( n|T| \) elements (and is endowed with the uniform measure), each player has \( A \) as his action space and there are \( n \) players with playoff function \( U(t_i) \) for all \( 1 \leq i \leq |T| \). We regard all such \( G_n \)'s equivalent and we refer to that equivalence class as the \( n^{th} \)-replica of \( G \).

Given a finite game as above, and the sequence \( \{G_n\} \) of its replicas, we can define a game with a continuum of players that can be thought of as the limit of \( \{G_n\} \). In such limit game, we can restrict players to play pure strategies and we still get a Nash equilibrium. We can then use such Nash equilibrium to construct an approximate equilibrium for the replica games in which a large fraction of players play pure strategies.

**Theorem 3** Let \( G = ((T, \nu), A, U) \) be a game with \( T \) finite and \( A = \Delta_m \) for some \( m \in \mathbb{N} \). Let \( G_n \) be its \( n^{th} \)-replica of \( G \). Then, for all \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that \( n \geq N \) implies that \( G_n \) has an \( \varepsilon \)-equilibrium in which all but a fraction of players smaller than \( \varepsilon \) plays a pure strategy.

Theorem 3 provides an example of a property which, because it holds for any game with a continuum of agents, will hold for sufficiently large finite games. Since it is in part a consequence of Theorem 1, it justifies the claim that the characterization given in Theorem 1 makes some asymptotic results available became quite natural.

It is also useful to compare Theorem 3 to Rashid [26]'s Theorem. Theorem 3 doesn't require that payoffs depend only on the average choice of the other players as in Rashid's, and so it more general in this respect. However, it requires replication, which is not required by Rashid's.\(^6\)

\(^6\)Note that, although not explicitly stated, Rashid’s Theorem requires that players payoff functions are selected from an equicontinuous family. For details, see Carmona [6].
6 Limit Points of Equilibria of Finite Games

6.1 Strict Equilibria

A natural question that arises from Theorem 1 is what kind of equilibria of a game with a continuum of players can be described as equilibria of a sequence of converging finite games. In other words, for what kind of equilibria can we take \( \varepsilon_n = 0 \) for all \( n \in \mathbb{N} \) in Theorem 1? Theorem 4 below provides a partial answer by showing that in games with a finite action space, strict equilibria satisfy that property.

A strategy \( f \) in a game \( G = ([0,1], \lambda, U, A) \) is a strict equilibrium if

\[
U(t)(f(t), \lambda \circ f^{-1}) > U(t)(a, \lambda \circ f^{-1})
\]

for all \( a \neq f(t) \) and a.e. \( t \in [0,1] \).

**Theorem 4** Let \( f \) be a strict equilibrium of a game \( G = ([0,1], \lambda, U, A) \) with \( A \) finite. Then, there exists a sequence \( \{\nu_n\}_{n=1}^{\infty} \) of measures such that:

1. \( \nu_n \) is the uniform measure on \( T_n \), a finite subset of \([0,1] \),
2. \( \nu_n \circ (U, f)^{-1} \Rightarrow \lambda \circ (U, f)^{-1} \),
3. \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \) and
4. \( f|_{T_n} \) is an equilibrium of \( G_n = ((T_n, \nu_n), U|_{T_n}, A) \) for all \( n \in \mathbb{N} \).

Thus, Theorem 4 shows that strict equilibria of games with a finite action space have the appealing property of being limit points of Nash equilibria of converging finite games. Unfortunately, not every game has a strict equilibrium. In what follows, we present an example of a game with a strict equilibrium.

Consider two bridges that start at the same location on the west bank but end in different places on the east bank. Denote this different location on the east bank by 0 and 1. There is a continuum of people living on the east bank, players being indexed by their address. That is, player \( t \in [0,1] \) lives at \( t \). Players preference on what bridge to take from the west bank depends on how many people is using each bridge, and on the distance from their end to their house.
Formally, we have a game with a continuum of players $G = ([0, 1], \lambda, U, A)$ where $A = \{0, 1\}$ and $0$ stands for taking the bridge that ends at $0$. We assume that 1. $U$ is continuous, 2. player 0 always prefers bridge 0 (i.e., $U(0)(0, \mu_A) > U(0)(1, \mu_A)$ for all $\mu_A$), 3. player 1 always prefers bridge 1, 4. $t \mapsto U(t)(0, \mu_A)$ is strictly decreasing for all $\mu_A$, 5. $t \mapsto U(t)(1, \mu_A)$ is strictly increasing for all $\mu_A$, 6. $U(t)(0, \cdot)$ is strictly decreasing in $\mu_A(\{0\})$ and 7. $U(t)(1, \cdot)$ is strictly increasing in $\mu_A(\{0\})$.

Given the above assumptions, for every $\mu_A$ there exists a unique $t \in (0, 1)$ such that $U(t)(0, \mu_A) = U(t)(1, \mu_A)$. Let $g : \mathcal{M} \to [0, 1]$ denote this dependence and, for every $t \in [0, 1]$, let $\varphi(t) \in \mathcal{M}$ be defined by $\varphi(t)(\{0\}) = t$. Then the unique fixed point $t^*$ of $t \mapsto g \circ \varphi(t)$ induces the unique Nash equilibrium $f$ of $G$: $f(t) = 0$ for $t < t^*$ and $f(t) = 1$ for $t > t^*$. Also, one easily sees that $f$ is a strict equilibrium.

### 6.2 Limit Equilibria

If $f$ is a strict equilibrium in a game $G = ([0, 1], \lambda, U, A)$ in which $A$ is finite, then $f$ is a limit equilibrium in the sense that it is the limit of a sequence of Nash equilibria of finite games, which converge to $G$ (in this case, every equilibrium in the sequence can be chosen to be $f$). This property is appealing insomuch we like to regard the ideal model of a game with a continuum of players as a limit model build upon the notion of games with finitely many players. Furthermore, we may hope that limit equilibria, despite being Nash equilibria of games in which players are strategically insignificant, will preserve the properties of equilibria of large finite games.\(^7\)

Thus, we are lead to use the above property in order to define a refinement of Nash equilibrium in games with a continuum of players as follows. A strategy $f$ in a game $G = ([0, 1], \lambda, U, A)$ is a limit equilibrium if there exists a sequence $G_n = ((T_n, \nu_n), U|_{T_n}, A)$ of games and a sequence of strategies $f_n : T_n \to A$ of $G_n$ such that:

1. $\nu_n$ is the uniform measure on $T_n$, a finite subset of $[0, 1]$,
2. $\nu_n \circ (U|_{T_n}, f_n)^{-1} \Rightarrow \lambda \circ (U, f)^{-1}$,
3. $\text{supp}(\nu_n) \to \text{supp}(\lambda)$ and

\(^7\)As the examples of section 6.3 illustrate, this may not be the case with any arbitrarily Nash equilibrium.
4. $f_n$ is an equilibrium of $G_n$ for all $n \in \mathbb{N}$.

It follows from Theorem 1 that any limit equilibrium is a Nash equilibrium.

**Theorem 5** Let $G = (([0,1], \lambda), A, U)$ be a game with a continuum of players and $f$ be a limit equilibrium of $G$. Then, $f$ is a Nash equilibrium of $G$.

Similarly, Theorem 4 shows that, in games with a finite action space, any strict equilibrium is a limit equilibrium. In particular, limit equilibria exist in games with a finite action space that have a strict equilibrium. However, if the action space is uncountable, then in general there is no Nash equilibria (see Khan, Rath and Sun [12]). In contrast, as Mas-Colell [20] as shown, an equilibrium distribution always exist. Hence, it is natural to start by asking when there exists an limit equilibrium distribution.

The definition of a limit equilibrium distribution is analogous to the one for limit equilibrium: A distribution $\tau$ on $U \times A$ satisfying $\tau \circ U^{-1} = \lambda$ is a limit equilibrium distribution for a game $G = (([0,1], \lambda), U)$ if there exists a sequence $G_n = (T_n, \nu_n), U|T_n, A)$ of games and a sequence of equilibrium distributions $\tau_n$ of $G_n$ such that:

1. $\nu_n$ is the uniform measure on $T_n$, a finite subset of $[0,1]$,
2. $\tau_n \Rightarrow \tau$, and
3. $\text{supp}(\nu_n) \rightarrow \text{supp}(\lambda)$.

Note that property 2 implies that $\nu_n \circ U^{-1} \Rightarrow \lambda$.

We first provide sufficient conditions for the existence of a limit equilibrium distribution (Theorem 6) and then conditions for the existence of a limit equilibrium strategy (Corollary 7). In both results, the assumptions of section 3 are still in place, i.e., $A$ is a compact metric space, and players have payoff function in $U$, the space of continuous function on $A \times M$.

**Theorem 6** Let $G = (([0,1], \lambda), U, A)$ be a game satisfying $A$ is a convex subset of a metric vector space and $a \mapsto U(t)(a, \tau)$ is quasi-concave for all $t \in [0,1]$ and all $\tau \in M$. Then $G$ has a limit equilibrium distribution.

We remark that the convexity of $A$ and the quasi-convexity of $a \mapsto U(t)(a, \tau)$ are needed to guarantee the existence of Nash equilibria in finite
games. Clearly, it can be replace a condition requiring that all finite game
with a sufficiently large set of players have a Nash equilibrium.\footnote{Formally, we require that there is \( N \in \mathbb{N} \) such that \( T \subseteq [0,1] \), finite and satisfying \( |T| \geq N \) implies that \( ((T, \nu), U_T, A) \) has a Nash equilibrium.} This may
be helpful since there are games that have equilibria if the set of players is
large, although they may fail to have equilibria if played by a small number
of players. An example of this case can be found in Bamón and Frayssé [2]'s
analysis of Cournot markets.

Alternatively, we could allow for mixed strategies in the finite games, and
so allow for a limit equilibrium \( f : [0,1] \to A \) to be the limit point of a
sequence of equilibria \( f_n : T_n \to \mathcal{M} \). Since \( A \) is homeomorphic to the closed
subset \( D = \{ p_a : a \in A \} \) of \( \mathcal{M} \), where \( p_a \) is the measure degenerated at \( a \), the
definition would still make sense. Then, in order to prove the existence of a
limit equilibrium (or, of a limit equilibrium distribution) it would be enough
to show the existence of Nash equilibria of the finite games in the converging
sequence with the property that only a vanishing fraction of players use mixed
strategies. Such result exists for \( \varepsilon \)-equilibrium, as shown by Rashid [26] and
Wooders, Selten and Cartwright [33], but, to our knowledge, not for Nash
equilibria.

We now turn to the question of existence of a limit equilibrium strategy.
Our result will be on the form of a representation result, that is, we ask
when is there a strategy that represents a given limit equilibrium distribu-
tion. A sufficient condition is provided in Theorem 7, which is an immediate
consequence of Lemma 6 in the Appendix.

**Theorem 7** Let \( G = ([0,1], \lambda), U, A \) be a game. Suppose that \( G \) has a
limit equilibrium distribution \( \tau \) satisfying \( \text{supp}(\tau) \) is countable. Then \( G \) has
a limit equilibrium.

### 6.3 Two Examples

In this section we present two examples of games in which the set of Nash
equilibria differs substantially from the set of limit equilibria.\footnote{These
examples originally appeared in Barlo and Carmona [3], although analyzed with
a different equilibrium concept.} We shall argue
that in all those games their unique limit equilibrium is the only plausible
Nash equilibrium.
6.3.1 A Game Solvable by Strict Dominance

Consider a game in which players have to choose a number between 1 and 10, its average being them given to the players. Formally, \( A = \{1, \ldots, 10\} \) and \( U(t) = u \) for all \( t \in [0, 1] \), where \( u(a, \tau) = \sum_{i=1}^{10} i \tau(\{i\}) \) for all \( a \in A \) and \( \tau \in \mathcal{M} \).

If this game is played by finitely many players, then clearly each will choose 10 — this is the unique Nash equilibrium of any finite version of the above game. It then follows that the strategy in which all players chose 10 is the unique limit equilibrium of the continuum game.\(^\text{10}^{\text{(1)}}\) In contrast, the set of its Nash equilibria equals the set of all strategies: since each player cannot influence the average when there are a continuum of players, he is indifferent between any action. In particular, a strategy in which every player chooses 1 is a Nash equilibrium, but an implausible outcome.

6.3.2 A Symmetric Cournot Market

Consider a Cournot market in which the price of the homogeneous good depends on the average (or per capita) supply \( p = 1 - \int_T f \, d\mu \) when the set of players is \( (T, \mu) \). Players are firms who choose a quantity in \([0, 2]\); all have a marginal cost of production of zero and no fixed cost. Hence, \( A = [0, 2] \) and \( U(t) = u \) for all \( t \in [0, 1] \), where \( u(a, \tau) = (1 - \int_A i \, d\tau) a \) and \( i : A \to A \) denotes the identity function.

Consider first the case in which there is a continuum of firms. The set of Nash equilibria in this game equals the set of all strategies \( f \) satisfying \( \int_{[0,1]} f \, d\lambda = 1 \). The reason is that \( \int_{[0,1]} f \, d\lambda = 1 \) implies \( p = 0 \), and so any firm is indifferent between any quantity to produce. In particular, we have asymmetric equilibria, such as \( f(t) = 0 \) for \( 0 \leq t \leq 1/2 \) and \( f(t) = 2 \) otherwise, in spite of the symmetry of the firms.

If we have a finite version of this game, where the set of players has \( n \in \mathbb{N} \) elements, there is a unique equilibrium, in which all firms produce \( n/(n+1) \) goods. This implies that there is a unique limit equilibrium as well: all firms produce 1. Thus, unlike for Nash equilibrium, the symmetry of the economic situation is preserved in the limit equilibrium.

\(^{10}\)As usual, two strategies that differ on a set of measure zero are considered the same.
7 Concluding Remarks

The main objective of this paper is to relate equilibria of games with a continuum of players with equilibria of games with finitely many players. This is done in two ways: First, by characterizing Nash equilibria of in terms of approximate equilibria of games with finitely many players. Second, by defining and proving the existence of equilibria of continuum games as limit points of equilibria of finite games.

Our characterization result shows that approximate equilibria of finite games provides an alternative way as to obtaining strategic insignificance of players, which is the main motivation of games with a continuum of players. In this way, they render as natural all the results that show that approximate equilibria of finite games have the same, or approximately the same, properties as equilibria of continuum games.

Our notion of limit equilibria defines an equilibrium concept in games with a continuum of players that refines Nash equilibrium. Its main objective is to eliminate Nash equilibria that can be thought as artifacts of the continuum model; indeed, for the examples presented in section 6.3 that seems to be the case. It is important to emphasize that players are still strategically insignificant in limit equilibrium as they are in Nash equilibrium. However, limit equilibria of a continuum game are those that can be related with Nash equilibria (in opposition of approximate equilibria) of finite games.

A Appendix

A.1 Lemmata

In this section, we present some technical results that we use. Lemma 5 below extend the well-know fact that the set of measures with finite support are dense in the set of all Borel measures in a separable metric space (see Parthasarathy [24, Theorem II.6.3]).

Lemma 5 Let $X$ be a separable metric space, $K \subseteq X$ be compact and $\mu \in \mathcal{M}(X)$. If $\mu = \lambda \circ h^{-1}$, where $h : [0, 1] \rightarrow X$ is measurable, and $\lambda$ is the Lebesgue measure on $[0, 1]$, then there exists a sequence $\{\mu_n\}$ in $\mathcal{M}(X)$ such that

1. $\mu_n \Rightarrow \mu$,
2. \( \lim_n \mu_n(K) = \mu(K) \)

3. for all \( n \in \mathbb{N} \), \( \mu_n = \nu_n \circ h_n^{-1} \) where \( T_n \) is a finite subset of \([0,1]\) and \( \nu_n \) is the uniform measure on \( T_n \) and 

4. \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \).

**Proof.** Let \( n \in \mathbb{N} \). Since \( K \) is compact, then we can write \( K = \bigcup_{i=1}^{J_n} B_{1/2n}(y_{n,j}) \) for some \( y_{n,j} \in K, j = 1, \ldots, J_n \). Hence, we can write \( K = \bigcup_{i=1}^{J_n} B_{n,j} \), where \( B_{n,1} = B_{1/2n}(y_{n,1}) \), and \( B_{n,j} = B_{1/2n}(y_{n,j}) \setminus \bigcup_{i=1}^{j-1} B_{1/2n}(y_{n,i}) \).

Thus, \( \{B_{n,j}\}_j \) is a disjoint collection, each of its member being a Borel set with a diameter no greater than \( 1/n \).

Since \( X \) is separable, we can write \( K^c = X \setminus K = \bigcup_{i=1}^\infty A_{n,i} \) where \( \{A_{n,i}\} \) is a disjoint collection, each of its member being a Borel set with a diameter no greater than \( 1/n \).

Let \( I_n \in \mathbb{N} \) be such that \( \sum_{i=I_n}^\infty \mu(A_{n,i}) < 1/n \). Let \( \{\tilde{q}_{n,j}\}_{j=1}^{I_n} \subset \mathbb{Q} \) and \( \{\tilde{p}_{n,i}\}_{i=1}^{I_n} \subset \mathbb{Q} \) be such that

\[
\begin{align*}
|\tilde{q}_{n,j} - \mu(B_{n,j})| &< 1/(nJ_n), j = 1, \ldots, J_n, \\
|\tilde{p}_{n,i} - \mu(A_{n,i})| &< 1/(nI_n), i = 1, \ldots, I_n - 1, \text{ and} \\
\sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} &= 1.
\end{align*}
\]

Also, if \( \mu(B_{n,j}) = 0, j = 1, \ldots, J_n \), let \( \tilde{q}_{n,j} = 0 \); if \( \mu(A_{n,i}) = 0, i = 1, \ldots, I_n - 1 \), let \( \tilde{p}_{n,i} = 0 \), and if \( \sum_{i=I_n}^\infty \mu(A_{n,i}) = 0 \) let \( \tilde{p}_{n,I_n} = 0 \). We remark that the above construction can always be done: if there is just one such set with positive measure this is clear as its measure will be 1, a rational number. If there are \( k > 1 \) such sets with positive measure, then approximate the measure of \( k - 1 \) of those sets by rational points in a way that the difference is smaller than \( \zeta > 0 \), where \( \zeta(J_n + I_n) < \min\{1/(nJ_n), 1/(nI_n)\} \), and set the rational approximation for the remaining set using the formula \( \sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1 \).

Since \( \sum_{i=I_n}^\infty \mu(A_{n,i}) = 1 - \sum_{j=1}^{J_n} \mu(B_{n,j}) - \sum_{i=1}^{I_n-1} \mu(A_{n,i}) \), it follows that \( |\tilde{p}_{n,I_n} - \sum_{i=I_n}^\infty \mu(A_{n,i})| < 2/n \) and that \( \sum_{j=1}^{J_n} \tilde{q}_{n,j} + \sum_{i=1}^{I_n} \tilde{p}_{n,i} = 1 \). Furthermore, there exists \( N_n \in \mathbb{N}, \{\tilde{q}_{n,j}\}_{j=1}^{J_n} \subset \mathbb{N} \) and \( \{\tilde{p}_{n,i}\}_{i=1}^{I_n} \subset \mathbb{N} \) such that \( \tilde{q}_{n,j}/N_n, j = 1, \ldots, J_n \) and \( \tilde{p}_{n,i}/N_n, i = 1, \ldots, I_n \).

Let \( i \in \{1, \ldots, I_n - 1\} \). Since \( \mu(A_{n,i}) = \lambda(\{t \in [0,1] : h(t) \in A_{n,i}\}) \), select \( p_{n,i} \) points from \( \{t \in [0,1] : h(t) \in A_{n,i}\} \). By the above convention,
\( p_{n,i} > 0 \) implies \( \mu(A_{n,i}) > 0 \), and so we can indeed select such points from \( \{ t \in [0, 1] : h(t) \in A_{n,i} \} \). Similarly, select \( p_{n,j} \) points from \( \{ t \in [0, 1] : h(t) \in \bigcup_{i=1}^{\infty} A_{n,i} \} \) and \( q_{n,j} \) points from \( \{ t \in [0, 1] : h(t) \in B_{n,j} \} \), \( j = 1, \ldots, J_n \). Let \( \{ t_{l_i}^{n,1} \}_{l_i=1}^{L_{n,1}} \) denote this collection of points from \([0, 1] \), where \( L_{n,1} = N_n \) and where \( t_1^{n,1} < \ldots < t_{L_{n,1}}^{n,1} \).

We want to define \( T_n \) in such a way that the distance between any two consecutive points is less than \( 1/n \), and also that the difference from the minimum (resp. maximum) point from zero (resp. 1) is less than \( 1/n \); this will imply \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \). We proceed by induction.

Let \( \mathcal{P} \) be the collection of those sets from

\[ \mathcal{N} = \{ A_{n,1}, \ldots, A_{n,J_n-1}, \bigcup_{i=1}^{\infty} A_{n,i}, B_{n,1}, \ldots, B_{n,J_n} \} \]

that have strictly positive measure. Suppose that \( \{ t_{l_i}^{n,k} \}_{l_i=1}^{L_{n,k}} \) is defined for \( k \in \mathbb{N} \) with \( t_1^{n,k} < \ldots < t_{L_{n,k}}^{n,k} \) and \( h(t_{l_i}^{n,k}) \) belongs to a set \( h^{-1}(P) \) with \( P \) in \( \mathcal{P} \) for all \( l \). Define \( t_0^{n,k} = 0 \), and \( t_{L_{n,k}+1}^{n,k} = 1 \). For \( l = 0, \ldots, L_{n,k} \), define

\[ t_l^{n,k} = \frac{t_{l+1}^{n,k} + t_{l+1}^{n,k}}{2} \]

if \( h(t_{l_i}^{n,k}) \) belongs to a set \( h^{-1}(P) \) with \( P \) in \( \mathcal{P} \); otherwise, let \( \tilde{t}_l^{n,k} \) be such that it belongs to a set \( h^{-1}(P) \) with \( P \) in \( \mathcal{P} \) and

\[ \left| t_l^{n,k} - \frac{t_{l+1}^{n,k} + t_{l+1}^{n,k}}{2} \right| < \frac{1}{2^k}. \]

Since the union of the sets in \( \mathcal{N} \) equals \( X \), then \( \{ h^{-1}(N) \}_{N \in \mathcal{N}} \) partitions \([0, 1] \). If \( N \in \mathcal{N} \) is such that \( (t_{l_i}^{n,k} + t_{l_i+1}^{n,k})/2 \in h^{-1}(N) \), then since \( \mu(N) = \lambda(h^{-1}(N)) \), it follows that any neighborhood of \( (t_{l_i}^{n,k} + t_{l_i+1}^{n,k})/2 \) intercepts a set \( h^{-1}(P) \) with \( P \in \mathcal{P} \); otherwise a neighborhood of \( (t_{l_i}^{n,k} + t_{l_i+1}^{n,k})/2 \) would be contained in \( \bigcup_{N \in \mathcal{N} \setminus \mathcal{P}} h^{-1}(N) \) implying that this set has positive Lebesgue measure — but this is impossible as \( \bigcup_{N \in \mathcal{N} \setminus \mathcal{P}} h^{-1}(N) \) is the union of finitely many sets of Lebesgue measure zero. Hence, we can construct \( \tilde{t}_l^{n,k} \) in the way described above.

Finally define \( \{ t_{l_i}^{n,k+1} \}_{l_i=1}^{L_{n,k+1}} \) as the union of \( \{ t_{l_i}^{n,k} \}_{l_i=1}^{L_{n,k}} \) and \( \{ \tilde{t}_l^{n,k} \}_{l=0}^{L_{n,k}} \) with \( t_1^{n,k+1} < \ldots < t_{L_{n,k+1}}^{n,k+1} \).

Let \( M_{n,k} \) be the maximum of \( t_{L_{n,k+1}+1}^{n,k+1} - t_{l_i}^{n,k} \), with \( t_0^{n,k+1} = 0 \), and \( t_{L_{n,k+1}+1}^{n,k+1} = 1 \). By construction, we have that

\[ M_{n,k+1} \leq \frac{M_{n,k}}{2} + \frac{1}{2^k}. \]
which implies, by induction, that

\[ M^{n,k+1} < \frac{M^{n,1}}{2^k} + \frac{k}{2^k}. \]  

(8)

Hence, \( \lim_{k \to \infty} M^{n,k} = 0 \), and so let \( K_n \) be such that \( M^{n,K_n} < 1/n \).

Let \( \gamma_n \in \mathbb{N} \) be such that

\[
\gamma_n \min \{ p_{n,i} : p_{n,i} > 0, \ i = 1, \ldots, I_n \} > |\{ t_{l_i}^{n,K_n} \}_{l=1}^{L_{n,K_n}} | \text{ and }
\gamma_n \min \{ q_{n,j} : q_{n,j} > 0, \ j = 1, \ldots, J_n \} > |\{ t_{l_i}^{n,K_n} \}_{l=1}^{L_{n,K_n}} |.
\]

(9)

Then select \( \gamma_n q_{n,j} \) points from \( \{ t \in [0,1] : h(t) \in B_{n,j} \} \), \( j = 1, \ldots, J_n \), \( \gamma_n p_{n,i} \) points from \( \{ t \in [0,1] : h(t) \in A_{n,i} \} \), \( i = 1, \ldots, I_n - 1 \), and \( \gamma_n p_{n,I_n} \) points from \( \{ t \in [0,1] : h(t) \in \bigcup_{i=I_n}^{\infty} A_{n,i} \} \), in a way that \( \{ t_{l_i}^{n,K_n} \}_{l=1}^{L_{n,K_n}} \) is contained in that collection of \( \gamma_n N_n \) points. This last requirement is possible to fulfil since \( \{ t_{l_i}^{n,K_n} \}_{l=1}^{L_{n,K_n}} \) is contained in \( \bigcup_{P \in \mathcal{P}} h^{-1}(P) \) and, for every \( P \in \mathcal{P} \), \( h^{-1}(P) \) is a set of positive Lebesgue measure, thus uncountable. Then we define \( T_n \) as the collection of these \( \gamma_n N_n \) points, and we define \( \mu_n = \nu_n \circ h_{|T_n}^{-1} \).

By construction, \( \sup_{z \in T_n} |t - z| < 1/n \) for all \( t \in [0,1] = \text{supp}(\lambda) \), and so \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \).

We claim that \( \lim_n \mu_n(K) = \mu(K) \). We have that

\[
\mu_n(K) = \nu_n \circ h_{|T_n}^{-1}(K)
= \nu_n(\{ t \in T_n : h(t) \in K \})
= \sum_{j=1}^{J_n} \nu_n(\{ t \in T_n : h(t) \in B_{n,j} \})
= \sum_{j=1}^{J_n} \gamma_n q_{n,j} = \sum_{j=1}^{J_n} \tilde{q}_{n,j}.
\]

(10)

Hence, \( |\mu_n(K) - \mu(K)| \leq \sum_{j=1}^{J_n} |\tilde{q}_{n,j} - \mu(B_{n,j})| < 1/n \to 0 \) as \( n \to \infty \).
Finally, we show that $\mu_n \Rightarrow \mu$. Let $g$ be a bounded, uniformly continuous real-valued function on $X$, and let $g$ be bounded by $M$. Since

$$\left| \int g d\mu_n - \int g d\mu \right| \leq \sum_{j=1}^{J_n} \left( \int_{B_{j,n}} g d\mu_n - \int_{B_{j,n}} g d\mu \right) + \sum_{i=1}^{I_n-1} \left( \int_{A_{i,n}} g d\mu_n - \int_{A_{i,n}} g d\mu \right) + \sum_{i=I_n}^{\infty} \left( \int_{A_{i,n}} g d\mu_n - \int_{A_{i,n}} g d\mu \right), \tag{11}$$

it is enough to show that each of the three terms on the right side of the above inequality converges to zero as $n$ converges to infinity.

We have that $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g d\mu_n \right| \leq M \sum_{i=I_n}^{\infty} \mu_n(A_{i,n}) = M \mu_n(\bigcup_{i=I_n}^{\infty} A_{i,n}) = M \bar{p}_{n,i} < M \left( \sum_{i=I_n}^{\infty} \mu(A_{i,n}) + 2/n \right) < 3M/n \to 0$ as $n \to \infty$. Similarly, $\left| \sum_{i=I_n}^{\infty} \int_{A_{i,n}} g d\mu \right| < M/\mu_i \to 0$ as $n \to \infty$. Hence

$$\left| \sum_{i=I_n}^{\infty} \left( \int_{A_{i,n}} g d\mu_n - \int_{A_{i,n}} g d\mu \right) \right| \to 0$$
as $n \to \infty$.

Let $\alpha_{n,j} = \inf_{x \in B_{n,j}} g(x)$ and $\beta_{n,j} = \sup_{x \in B_{n,j}} g(x)$, $j = 1, \ldots, J_n$. Also, let $\alpha_{n,i} = \inf_{x \in A_{n,i}} g(x)$ and $\beta_{n,i} = \sup_{x \in A_{n,i}} g(x)$, $i = 1, \ldots, I_n - 1$. Since $g$ is uniformly continuous, and the diameter of $A_{n,i}$, and $B_{n,j}$ converge to zero as $n$ converges to infinity uniformly on $i$ and $j$ respectively, it follows that $\sup_{j}(\beta_{n,j} - \alpha_{n,j})$ and $\sup_{j}(\beta_{n,j} - \alpha_{n,j})$ converge to zero as $n$ converges to infinity.

Let $j \in \{1, \ldots, J_n\}$, and let $\{x_{n,j}^m\}_{m=1}^{\gamma_{n} q_{n,j}} = h(T_n) \cap B_{n,j}$. Then

$$\int_{B_{n,j}} g d\mu_n = \sum_{m=1}^{\gamma_{n} q_{n,j}} g(x_{n,j}^m). \tag{12}$$

We have that

$$\int_{B_{n,j}} g d\mu - \sum_{m=1}^{\gamma_{n} q_{n,j}} g(x_{n,j}^m) = \beta_{n,j} \mu(B_{n,j}) - \alpha_{n,j} \frac{\gamma_{n} q_{n,j}}{\gamma_n N_n} \leq \beta_{n,j} \mu(B_{n,j}) - \alpha_{n,j} \frac{\gamma_{n} q_{n,j}}{\gamma_n N_n}$$

$$= \mu(B_{n,j})(\beta_{n,j} - \alpha_{n,j}) + \alpha_{n,j}(\mu(B_{n,j}) - q_{n,j}) \tag{13}$$

$$\leq \mu(B_{n,j}) \sup_{j'}(\beta_{n,j'} - \alpha_{n,j'}) + M \left| \mu(B_{n,j}) - q_{n,j} \right|. \tag{13}$$
Similarly, 
\[ \sum_{m=1}^{\gamma_n q_n} g(x^m_{n,j}) - \int_{B_{n,j}} g \, d\mu \leq \mu(B_{n,j}) \sup_{j'} (\beta_{n,j'} - \alpha_{n,j'}) + M |\mu(B_{n,j}) - \bar{q}_{n,j}|. \]

Thus, 
\[ \left| \int_{B_{j,n}} g \, d\mu - \int_{B_{j,n}} g \, d\mu \right| < \mu(B_{n,j}) \sup_{j'} (\beta_{n,j'} - \alpha_{n,j'}) + \frac{M}{nJ_n}, \]

and so 
\[ \left| \sum_{j=1}^{J_n} \left( \int_{B_{j,n}} g \, d\mu - \int_{B_{j,n}} g \, d\mu \right) \right| < \sup_{j} (\beta_{n,j} - \alpha_{n,j}) + \frac{M}{n} \to 0 \quad (14) \]
as \( n \to \infty \).

Similarly, we can show that 
\[ \left| \sum_{i=1}^{J_n} \left( \int_{A_{i,n}} g \, d\mu - \int_{A_{i,n}} g \, d\mu \right) \right| < \sup_{i} (\beta_{n,i} - \alpha_{n,i}) + \frac{M}{n} \to 0 \]
as \( n \to \infty \). Hence, we conclude that \( |\int g \, d\mu_n - \int g \, d\mu| \to 0 \) as \( n \to \infty \) and, since \( g \) is arbitrary, that \( \mu_n \Rightarrow \mu \). \( \blacksquare \)

The following result, Lemma 6, is a representation theorem for measures with a countable support. In the particular context of a game with a continuum of players, it says that any distribution with a countable support can be thought of as the distribution induced by a strategy together with the function assigning preferences to players.

**Lemma 6** Let \( \mu \) be a distribution on \( U \times A \) satisfying \( \mu_U = \lambda \circ U^{-1} \), where \( U : [0, 1] \to U \) is measurable and \( \lambda \) denotes the Lebesgue measure on \([0, 1]\).

If \( \text{supp}(\mu) \) is countable, then there exists a measurable function \( f : [0, 1] \to A \) such that \( \mu = \lambda \circ (U, f)^{-1} \).

**Proof.** Let \( \{a_i\}_{i=1}^{\infty} \subseteq A \) and \( \{b_j\}_{j=1}^{\infty} \subseteq U \) be such that \( \text{supp}(\mu) \subseteq \{a_i\}_i \times \{b_j\}_j \).

Let \( j \in \mathbb{N} \) be fixed. We have that 
\[ \sum_{i=1}^{\infty} \mu(\{b_j\} \times \{a_i\}) = \mu_U(\{b_j\}) = \lambda(U^{-1}(\{b_j\})). \quad (15) \]
In particular, \( 0 \leq \mu(\{b_j\} \times \{a_i\}) \leq \lambda(U^{-1}(\{b_j\})) \). Hence, by Liapunov’s Theorem (see Hildenbrand [17, prop. 15, p. 45]), there exists a Borel measurable set \( \tilde{T}_{j,1} \subseteq U^{-1}(\{b_j\}) \) satisfying \( \lambda(\tilde{T}_{j,1}) = \mu(\{b_j\} \times \{a_1\}) \). By induction, we can find a partition \( \{T_{j,i}\}_{i=1}^{\infty} \) of \( U^{-1}(\{b_j\}) \) satisfying \( T_{j,i} \) is Borel measurable and \( \lambda(T_{j,i}) = \mu(\{b_j\} \times \{a_i\}) \) for all \( i \): if \( k > 1 \) and we have constructed a disjoint collection \( \{\tilde{T}_{j,i}\}_{i=1}^{k-1} \) of Borel measurable sets, we find \( \tilde{T}_{j,k} \subseteq U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{k-1} \tilde{T}_{j,i} \) satisfying \( \lambda(\tilde{T}_{j,k}) = \mu(\{b_j\} \times \{a_k\}) \), which is possible by Liapunov’s Theorem since

\[
\lambda(U^{-1}(\{b_j\})) = \sum_{i=1}^{\infty} \mu(\{b_j\} \times \{a_i\}) \geq \sum_{i=1}^{k} \mu(\{b_j\} \times \{a_i\})
\]

implies that

\[
\lambda(U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{k-1} \tilde{T}_{j,i}) = \lambda(U^{-1}(\{b_j\})) - \sum_{i=1}^{k-1} \lambda(\tilde{T}_{j,i}) = \lambda(U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{k-1} \tilde{T}_{j,i}) - \sum_{i=1}^{k-1} \mu(\{b_j\} \times \{a_i\}) \geq \mu(\{b_j\} \times \{a_k\}).
\]

Clearly, \( \lambda(\cup_{i=1}^{\infty} \tilde{T}_{j,i}) = \lambda(U^{-1}(\{b_j\})) \), and so define \( T_{j,i} = \tilde{T}_{j,i} \) for \( i > 1 \) and \( T_{j,1} = \tilde{T}_{j,1} \cup \left(U^{-1}(\{b_j\}) \setminus \cup_{i=1}^{\infty} \tilde{T}_{j,i}\right) \).

Then, we define \( f : [0,1] \to A \) by \( f(t) = a_i \) if \( t \in T_{j,i} \) for some \( j \in \mathbb{N} \) and \( i \in \mathbb{N} \). For \( i, j \in \mathbb{N} \), we have that \( \lambda \circ (U,f)^{-1}(\{b_j\} \times \{a_i\}) = \lambda(T_{j,i}) = \mu(\{b_j\} \times \{a_i\}) \). This implies that \( \lambda \circ (U,f)^{-1} = \mu \) as follows: Denote by

\[
C = \bigcup_{i,j \in \mathbb{N}} \{(b_j, a_i)\}.
\]

Then, \( \mu(C^c) = \lambda \circ (U,f)^{-1}(C^c) = 0 \). If \( B \) is a Borel set in \( U \times A \), and if \( L = \{(j,i) \in \mathbb{N} \times \mathbb{N} : (b_j, a_i) \in B\} \), then

\[
\mu(B) = \sum_{(j,i) \in L} \mu(\{b_j\} \times \{a_i\}) = \sum_{(j,i) \in L} \lambda \circ (U,f)^{-1}(\{b_j\} \times \{a_i\}) = \lambda \circ (U,f)^{-1}(B).
\]
In this section we present the proofs of all the results stated in the main text.

A.2 Proofs

In this section we present the proofs of all the results stated in the main text.

Proof of Lemma 1. Let \( \{r_k\}_{k=1}^{\infty} \subseteq A \) be dense in \( A \). Then, the continuity of \( U(t) \) for all \( t \in [0, 1] \), implies that

\[
E(f, \varepsilon, \lambda) = \bigcap_{k} \{ t \in [0, 1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \}. \tag{20}
\]

Hence, it is enough to show that \( \{ t \in [0, 1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \} \) is measurable for all \( k \).

Let \( k \in \mathbb{N} \), and \( \eta > 0 \). By remark 7 in Rath [27], \( (t, a) \mapsto U(t)(a, \lambda \circ f^{-1}) \) is measurable. Denote this function by \( \hat{u} \). By changing it in a set of measure zero, we may assume that it is Borel measurable; similarly, assume that \( f \) is Borel measurable. Then, the functions \( t \mapsto U(t)(f(t), \lambda \circ f^{-1}) \) and \( t \mapsto U(t)(r_k, \lambda \circ f^{-1}) \) are Borel measurable in \( F \) since they equal \( \hat{u} \circ f \), and \( \hat{u} \circ g \) respectively, where \( f(t) = (t, f(t)) \), and \( g(t) = (t, r_k) \) are measurable. Thus, by Lusin’s Theorem, let \( C \subseteq [0, 1] \) be a compact set, \( \lambda([0, 1] \setminus C) < \eta \), be such that \( t \mapsto U(t)(f(t), \lambda \circ f^{-1}) \) and \( t \mapsto U(t)(r_k, \lambda \circ f^{-1}) \) are continuous in \( C \). Since

\[
\{ t \in [0, 1] : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \} \setminus \{ t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \}
\]

the outer measure of \( \{ t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \} \setminus \{ t \in C : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(r_k, \lambda \circ f^{-1}) - \varepsilon \} \) is smaller than \( \eta \). Hence, it is enough to show that \( \{ t \in C : u(t, f(t), \lambda \circ f^{-1}) \geq u(t, r_k, \lambda \circ f^{-1}) - \varepsilon \} \) is closed (see Wheeden and Zygmund [32, Lemma 3.22]). This follows easily from the fact that both \( t \mapsto U(t)(f(t), \lambda \circ f^{-1}) \) and \( t \mapsto U(t)(r_k, \lambda \circ f^{-1}) \) are continuous in \( C \).
Proof of Lemma 2. For notational convenience let \( h = (U, f) \). We have

\[
  h^{-1}(B_\tau) = \\
  \{ t : (U(t), f(t)) \in B_\tau \} = \\
  \{ t : U(t)(f(t), \tau_A) \geq U(t)(A, \tau_A) \} = \\
  \{ t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1}) \}. 
\]

Hence \( \tau \) is an equilibrium distribution if and only if \( \tau(\{(u, a) : u(a, \tau_A) \geq u(A, \tau_A)\}) = 1 \) if and only if \( \lambda(h^{-1}(B_\tau)) = 1 \) if and only if \( \lambda(\{ t : U(t)(f(t), \lambda \circ f^{-1}) \geq U(t)(A, \lambda \circ f^{-1}) \}) = 1 \) if and only if \( f \) is a Nash equilibrium. ■

Proof of Lemma 3. Let \( \tau = \nu \circ f^{-1} \) and \( \tau' = \nu \circ g^{-1} \). We have that

\[
  \tau(\{a\}) = \frac{\{ t \in T : f(t) = a \}}{|T|} 
\]

for all \( a \in A \), and similarly for \( \tau' \). Since there is just one \( t \in T \) such that \( g \) and \( f \) differ, there are only two points in \( A \) such that \( \tau(\{a\}) \) differ from \( \tau'(\{a\}) \). Note that in that case \( |\tau(\{a\}) - \tau'(\{a\})| = 1/|T| \). Denoting this points \( a' \) and \( a'' \), we see that for any Borel subset \( E \) of \( A \) \( |\tau(E) - \tau'(E)| = 0 \) if either \( a' \) and \( a'' \) belong to \( E \) or \( a' \) and \( a'' \) belong to \( E^c \), while \( |\tau(E) - \tau'(E)| = 1/|T| \) otherwise. This implies \( \rho(\tau, \tau') \leq 1/|T| \). ■

Proof of Lemma 4. For notational convenience let \( h = (U, f) \). We have

\[
  h^{-1}(B_\tau^\varepsilon) = \\
  \{ t \in T : (U(t), f(t)) \in B_\tau^\varepsilon \} = \\
  \{ t \in T : U(t)(f(t), \tau_A) \geq U(t)(a', \tau_{A}^{u,a'}) - \varepsilon \text{ for all } a' \} = \\
  \{ t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq U(t)(a', \nu \circ (f \setminus a')^{-1}) - \varepsilon \text{ for all } a' \}. 
\]

Hence, \( \tau \) is an \( \varepsilon \)-equilibrium distribution if and only if \( \tau(B_\tau^\varepsilon) \geq 1 - \varepsilon \) if and only if \( \nu(h^{-1}(B_\tau^\varepsilon)) \geq 1 - \varepsilon \) if and only if \( \nu(\{ t \in T : U(t)(f(t), \nu \circ f^{-1}) \geq U(t)(a', \nu \circ (f \setminus a')^{-1}) - \varepsilon \text{ for all } a' \}) \geq 1 - \varepsilon \) if and only if \( f \) is an \( \varepsilon \)-equilibrium. ■

Proof of Theorem 1. (Necessity) Let \( \tau = \lambda \circ (U, f)^{-1} \); by Lemma 2 \( \tau \) is an equilibrium distribution on \( \mathcal{U} \times A \). For notational convenience,
let \( h = (U, f) \). Let \( n \in \mathbb{N} \) and define \( \varepsilon_n = 1/n \). Since \( \mathcal{U} \) is a complete separable metric space, and \( A \) is compact then \( \tau \) is tight by Parthasarathy [24, Theorem II.3.2], as \( \mathcal{U} \times A \) is also a complete separable metric space.

Since \( B_\tau \) is closed, and so a Borel set, let \( K_n \subseteq B_\tau \) be compact, and satisfy \( \tau(B_\tau \setminus K_n) < 1/2n \). Since \( \tau \) is an equilibrium distribution, it follows that \( \tau(B_\tau) = 1 \), and so \( \tau(K_n) > 1 - 1/2n \). If \( \pi \) denotes the projection of \( \mathcal{U} \times A \) into \( \mathcal{U} \), then \( \pi(K_n) \) is compact, and \( K_n \subseteq \pi(K_n) \times A \). In particular, \( \pi(K_n) \) is equicontinuous by the Ascoli-Arzela Theorem since \( A \), and \( \mathcal{M} \) are both compact metric spaces. Furthermore, denoting \( C_n = \pi(K_n) \times A \), it follows \( \tau(C_n \cap B_\tau) \geq \tau(K_n \cap B_\tau) = \tau(K_n) > 1 - 1/2n \).

Let \( \delta_n > 0 \) be such that \( d((a, \mu), (b, \nu)) := \max\{d_A(a, b), \rho(\mu, \nu)\} < \delta_n \) implies that \( |u(a, \mu) - u(b, \nu)| < 1/4n \) for all \( u \in \pi(K_n) \). By Lemma 5, there exists a sequence \( \{\mu_j\} \) such that \( \mu_j \Rightarrow \tau \), \( \lim_j \mu_j(C_n \cap B_\tau) = \tau(C_n \cap B_\tau) \), \( \mu_j = \nu_j \circ h_T^{-1} \) where \( \nu_j \) is the uniform measure on some finite set \( T_j \subset [0, 1] \), and \( T_j = \text{supp}(\nu_n) \rightarrow \text{supp}(\lambda) = [0, 1] \). Hence, \( \mu_{A,j} \Rightarrow \tau_A \), and let \( J_n \in \mathbb{N} \) be such that \( \rho(\mu_{A,n}, \tau_A) < \delta_n \), \( |\mu_{n}(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n \), \( \rho(\mu_{A,n}, \tau) < 1/n \) and \( 1/|T_{J_n}| < \delta_n \). Define \( \tau_n = \mu_{J_n} \), \( T_n = T_{J_n} \) and \( \nu_n = \nu_{J_n} \).

By construction of \( \{\tau_n\} \) we have \( \tau_n \Rightarrow \tau \), and that, for every \( n \in \mathbb{N} \), \( \rho(\tau_n, \tau) < 1/n \), \( \rho(\tau_{A,n}, \tau_A) < \delta_n \), \( |\tau_n(C_n \cap B_\tau) - \tau(C_n \cap B_\tau)| < 1/2n \), \( 1/|T_n| < \delta_n \) and \( \tau_n = \nu_n \circ h_T^{-1} \) where \( T_n \) is a finite subset of \([0, 1]\) and \( \nu_n \) is the uniform measure on \( T_n \).

We have that \( C_n \cap B_\tau \subseteq C_n \cap B_{1/n} \), since if \( (u, a) \in C_n \cap B_\tau \) and \( a' \in A \) then \( u(a, \tau_{A,n}) > u(a, \tau_{A}) - 1/4n \geq u(a', \tau_{A}) - 1/4n > u(a', \tau_{A,n}) - 1/2n > u(a', \tau_{A,n}) - 1/n \) since \( \rho(\tau_{A,n}, \tau_A) < \delta_n \) and \( \rho(\tau_{A,n}, \tau_{A,n}) \leq 1/|T_n| < \delta_n \) by Lemma 3. So \( 1 - 1/n < \tau(C_n \cap B_\tau) - 1/2n < \tau_n(C_n \cap B_\tau) \leq \tau_n(C_n \cap B_{1/n}) \leq \tau_n(B_{1/n}) \). Hence, \( \tau_n = \nu_n \circ h_T^{-1} = \nu_n \circ (U_{T_n}, f_{T_n})^{-1} \) is an \( \varepsilon_n \)-equilibrium distribution of the game \( \tau_{A,n} = \nu_n \circ U_{T_n}^{-1} \). By Lemma 4 then \( f_{T_n} \) is an \( \varepsilon_n \)-equilibrium of \( G_n = ((T_n, \nu_n), U_{T_n}) \).

(Sufficiency) Let \( \tau = \lambda \circ (U, f)^{-1} \) and let \( \tau_n \Rightarrow \tau \), where \( \tau_n = \nu_n \circ (U, f)^{-1} \) is an \( \varepsilon_n \)-equilibrium distribution and \( \varepsilon_n \searrow 0 \). Then \( \tau_{A,n} \Rightarrow \tau_A \) and \( \text{supp}(\nu_n) \rightarrow \text{supp}(\lambda) \); so, taking a subsequence if necessary, we may assume that \( \rho(\tau_A, \tau_{A,n}) < 1/2n \) and \( \rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) < 1/2n \) for every \( u \in \mathcal{U} \) and \( a, a' \in A \); the second inequality is obtained via Lemma 3 by taking \( 1/|\text{supp}(\nu_n)| < 1/2n \). Clearly, we have \( \rho(\tau_A, \tau_{A,n}^{u,a,a'}) < 1/n \) for every \( u \in \mathcal{U} \) and \( a, a' \in A \).
Define, for each \( u \in U \),
\[
\beta_n(u) = \sup_{a \in A, \nu \in \mathcal{M}} \{|u(a, \nu) - u(a, \tau_A)| : d(\nu, \tau_A) < 1/n\}.
\]
Since \( u \) is continuous on \( A \times \mathcal{M} \), which is compact, it follows that \( u \) is uniformly continuous. Thus, \( \beta_n(u) \downarrow 0 \) as \( n \to \infty \). We claim that \( \beta_n \) is continuous in \( U \).

Let \( \eta > 0 \). Define \( \delta = \eta / 2 \). Then if \( \|u - v\| < \delta \), we have for any \( a \in A \), and \( \nu \in \mathcal{M} \) such that \( d(\nu, \tau_A) < 1/n \)
\[
|v(a, \nu) - v(a, \tau_A)| \leq |v(a, \nu) - u(a, \nu)| + |u(a, \nu) + u(a, \tau_A)| + |v(a, \tau_A) - u(a, \tau_A)| < \delta + \beta_n(u) + \delta,
\]
and so \( \beta_n(v) \leq 2\delta + \beta_n(u) < \eta + \beta_n(u) \). By symmetry, \( \beta_n(u) < \eta + \beta_n(v) \), and so \( |\beta_n(u) - \beta_n(v)| < \eta \). Hence, \( \beta_n \) is continuous as claimed.

Given the definition of \( \beta_n \), we have that \( B_{\tau_n}^{\varepsilon_n} \subseteq D_n := \{(u, a) : u(a, \tau_A) \geq u(A, \tau_A) - \varepsilon_n - 2\beta_n(u)\} \). Since \( \beta_n \) is continuous, we see that \( D_n \) is closed, and so Borel measurable. Thus, \( \tau_n(D_n) \geq 1 - \varepsilon_n \). Also, \( D_n \searrow B_{\tau} \). Hence, for fixed \( j \in \mathbb{N}, j \geq n \), it follows that \( \tau_j(D_n) \geq \tau_j(D_j) \geq 1 - \varepsilon_j \geq 1 - \varepsilon_n \), and so \( \tau(D_n) \geq \lim \sup_j \tau_j(D_n) \geq 1 - \varepsilon_n \). Hence, \( \tau(B_{\tau}) = \lim_n \tau(D_n) = 1 \). Therefore, \( \tau = \lambda \circ (U, f)^{-1} \) is an equilibrium distribution of \( \lambda \circ U^{-1} \) and so \( f \) is an equilibrium of \( G \). \( \blacksquare \)

We remark that Theorem 1 still holds if we replace the condition \( \varepsilon_n \searrow 0 \) by the weaker condition \( \varepsilon_n \to 0 \). Clearly, we only need to show how the proof of the sufficiency part changes. The condition \( \varepsilon_n \searrow 0 \) was only used to show that \( \lim_n \tau(D_n) = 1 \); we will now show that this continues to hold even when we only have \( \varepsilon_n \to 0 \).

Note that \( D_n \) decreases (to \( B_{\tau} \)). Let \( \gamma > 0 \) be given and let \( j \in \mathbb{N} \) be such that \( j \geq n \) and \( \varepsilon_k < \gamma \) whenever \( k \geq j \). Then, if \( k \geq j \), it follows that \( \tau_k(D_n) \geq \tau_k(D_k) \geq 1 - \varepsilon_k \geq 1 - \gamma \), and so \( \tau(D_n) \geq \lim \sup_j \tau_j(D_n) \geq 1 - \gamma \). Hence, \( \lim_n \tau(D_n) = 1 \).

**Proof of Theorem 2.** (Sufficiency) By Theorem 1 and the above remark, \( g \) is a Nash equilibrium of \( G \), and since \( g \sim f \), so is \( f \).

(Necessity) Define \( g \) by changing \( f \) only for those players that are not optimizing: if \( t \in [0, 1] \) is such that there exists \( a^* \in A \) satisfying \( U(t)(f(t), \lambda \circ f^{-1}) < U(t)(a^*, \lambda \circ f^{-1}) \) and \( U(t)(A, \lambda \circ f^{-1}) \leq U(t)(a^*, \lambda \circ f^{-1}) \), then define
$g(t) = a^*$. Since $f$ is a Nash equilibrium of $G$, it follows that $f \sim g$ and that $g$ is a Nash equilibrium.

Let $\tau = \lambda \circ (U, g)^{-1}$ and let $\tau_n = \nu_n \circ (U|_{T_n}, g|_{T_n})^{-1}$. Define

$$\alpha_n(u) = \sup_{a \in A, \nu \in M} \{ |u(a, \mu) - u(a, \phi)| : \rho(\mu, \phi) \leq \rho(\tau_{A,n}, \tau_A) \},$$

and $\alpha_n = \sup_{u \in U([0,1])} \alpha_n(u)$; similarly, define

$$\gamma_n(u) = \sup_{a \in A, \nu \in M} \{ |u(a, \mu) - u(a, \phi)| : \rho(\mu, \phi) \leq \rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) \},$$

and $\gamma_n = \sup_{u \in U([0,1])} \gamma_n(u)$.

As in the proof of Theorem 1 we can show that $B_\tau \subseteq B_{\tau_n}^{\gamma_n}$ if we define $\varepsilon_n = 2\alpha_n + \gamma_n$. Since $U([0,1])$ is equicontinuous and $\lim_{n \to \infty} \rho(\tau_{A,n}, \tau_A) = \rho(\tau_{A,n}, \tau_{A,n}^{u,a,a'}) = 0$, it follows that $\varepsilon_n \to 0$. Finally, note that $B_\tau = (U, g)([0,1])$ since all players are optimizing by choosing according to $g$; hence, $\tau_n(B_{\tau_n}^{\varepsilon_n}) = 1$, and so $g|_{T_n}$ is an $\varepsilon_n$–equilibrium of $G_n$. ■

Proof of Remark 2. For each $n \in \mathbb{N}$, we define $\gamma_n = \inf \{ \varepsilon \geq 0 : f|_{T_n} \text{ is an } \varepsilon - \text{equilibrium of } G_n \}$. Note that the set $\{ \varepsilon \geq 0 : f|_{T_n} \text{ is an } \varepsilon - \text{equilibrium of } G_n \}$ is nonempty since if $B > 0$ is such that $u$ is bounded by $B$ for all $u \in U([0,1])$, then $f|_{T_n}$ is an $2B$–equilibrium of $G_n$. Define $\varepsilon_n = \gamma_n + 1/n$. Thus, it is enough to show that $\gamma_n \to 0$.

Let $\eta > 0$ be given. Denote $\tau_n = \nu_n \circ (U|_{T_n}, f|_{T_n})^{-1}$ and $\tau = \lambda \circ (U, f)^{-1}$. Let $\delta > 0$ be such that $d((a, \mu), (b, \nu)) := \max \{ d_A(a, b), \rho(\mu, \nu) \} < \delta$ implies that $|u(a, \mu) - u(b, \nu)| < \eta/4$ for all $u \in U([0,1])$. Finally, let $N \in \mathbb{N}$ be such that $n \geq N$ implies that $1/|T_n| < \delta$, $\rho(\tau_{A,n}, \tau_A) < \delta$ and $|\tau_n(B_\tau) - \tau(B_\tau)| < \eta$. This last inequality follows from the fact that $\text{supp}(\tau)$ is finite and $\text{supp}(\tau_n) \subseteq \text{supp}(\tau)$.

Let $n \geq N$. As in the proof of Theorem 1, we can show that $B_\tau \subseteq B_{\tau_n}^n$. Hence,

$$1 - \eta = \tau(B_\tau) - \eta < \tau_n(B_\tau) \leq \tau_n(B_{\tau_n}^n), \quad (25)$$

and $f|_{T_n}$ is an $\eta$–equilibrium of $G_n$. This implies that $\gamma_n \leq \eta$ and, since $\eta$ is arbitrary, that $\gamma_n \to 0$. ■

Proof of Theorem 3. Let $T = \{ t_1, \ldots, t_k \}$ and $u_i = U(t_i)$ for all $1 \leq i \leq k$. Note first that the names of the players are irrelevant in the following sense: if the conclusion of Theorem 3 holds for a game $\hat{G} = ((\hat{T}, \nu), A, U)$,
with $|\hat{T}| = k$ and $U(\hat{t}_i) = u_i$ for all $1 \leq i \leq k$, then it will hold for $G$. In words, we are free to choose the name of the players.

Let $P$ denote the set of vertices of $A = \Delta_m$. Consider the game $G_\lambda = \left(\{[0, 1], \lambda\}, P, V\right)$ where $V(t) = u_t$ if $t \in T_i = \left[\frac{i-1}{k}, \frac{i}{k}\right)$ for $1 \leq i \leq k - 1$ and $V(t) = u_k$ if $t \in T_k = \left[\frac{k-1}{k}, 1\right]$. Then $G_\lambda$ has Nash equilibrium $f$ in which all players optimize (this follows by Theorem 1 in Mas-Colell [21] and by Lemma 6 above; see also Khan and Sun [13, p. 9]). By the expected utility hypothesis $(a = \sum_{i=1}^n a_i e_i$ implies that $U(t)(a, \mu) = \sum_{i=1}^n a_i U(t)(e_i, \mu))$, $f$ is also a Nash equilibrium of the game $\left(\{[0, 1], \lambda\}, A, V\right)$. By Lusin’s Theorem (see Wheeden and Zygmund [32, Theorem 4.20]) and Tietze-Urysohn’s Extension Theorem (see Machado [19, 1.9.5, p. 110]\(^{11}\)), let for every $l \in \mathbb{N}$, $g_l : [0, 1] \to A$ be continuous and satisfy $\lambda\left(\{g_l \neq f\}\right) < 1/l$. Then, $\lambda \circ g_l^{-1} \Rightarrow \lambda \circ f^{-1}$. So for every $\gamma > 0$ we can find $l$ such that $\sup_{a \in A} |u_t(a, \lambda \circ g_l^{-1}) - u_t(a, \lambda \circ f^{-1})| < \gamma$ and $\lambda\left(\{g_l \neq f\}\right) < \gamma$, which implies that $g_l$ is a $\gamma$-equilibrium. Therefore, we have shown that there exists a continuous function $g : [0, 1] \to A$ satisfying $\lambda\left(\{g \neq f\}\right) < \varepsilon/2$ and $g$ is an $\varepsilon/2$-equilibrium.

For every $n \in \mathbb{N}$, partition $[0, 1]$ in $nk$ intervals of the same length:

$$
[0, 1] = \left[0, \frac{1}{nk}\right] \cup \left[\frac{1}{nk}, \frac{2}{nk}\right] \cup \ldots \cup \left[\frac{nk-1}{nk}, 1\right].
$$

Let $T_{j,n} = \left[\frac{j-1}{nk}, \frac{j}{nk}\right)$ and select a point $t_{j,n} \in T_{j,n}$ in the following way: if $T_{j,n} \cap \{g = f\} \neq \emptyset$ then let $t_{j,n} \in T_{j,n} \cap \{g = f\}$. Let $\nu_n$ be the uniform measure with support equal to $\{t_{j,n}\}_{j=1}^{nk}$. It follows that $\nu_n\left(\{g \neq f\}\right) < \varepsilon/2$ since otherwise we would be able to find $p$ sets $\{T_{j,i,n}\}_{i=1}^p$ such that $T_{j,i,n} \subseteq \{g \neq f\}$, with $p/nk \geq \varepsilon/2$; this would imply that $\lambda\left(\{g \neq f\}\right) \geq \lambda(\bigcup_{i=1}^p T_{j,i,n}) = p/nk \geq \varepsilon/2$, a contradiction. We also have that $\nu_n \Rightarrow \lambda$ by construction. Furthermore, if $h : T_i \to \mathbb{R}$ is bounded and uniformly continuous then

$$
\lim_{n \to \infty} \int_{T_i} hd\nu_n = \int_{T_i} hd\lambda,
$$

(26)

for all $1 \leq i \leq k$. We finally define $G_n = (\{t_{j,n}\}_{j=1}^{nk}, \nu_n, A, V_n)$, where $V_n$ is the restriction of $V$ to $\{t_{j,n}\}_{j=1}^{nk}$. Note that $G_n$ is a replica of $G$ because of the way $V$ was defined.

\(^{11}\)The particular statement we need for our purposes is: if $F \subseteq [0, 1]$ is closed and $h : F \to \Delta_m$ is continuous then there exists a continuous function $\tilde{h} : [0, 1] \to \Delta_m$ such that $\tilde{h}(t) = h(t)$ for all $t \in F$. 

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Let \( g_n \) denote the restriction of \( g \) to \( \{t_{j,n}\}_{j=1}^{nk} \). We claim that \( \nu_n \circ (V_n, g_n)^{-1} \Rightarrow \lambda \circ (V, g)^{-1} \). Let \( h : U \times A \rightarrow \mathbb{R} \) be a uniformly continuous function. Then, by the change-of-variable-formula (see Hildenbrand [17, prop. 36, p.50]),

\[
\int_{U \times A} h d\nu_n \circ (V_n, g_n)^{-1} = \int_{[0,1]} h \circ (V, g) d\nu_n = \sum_{i=1}^{k} \int_{T_i} h \circ (V, g) d\nu_n = \sum_{i=1}^{k} \int_{T_i} h \circ (V, g) d\lambda = \int_{[0,1]} h \circ (V, g) d\lambda = \int_{U \times A} h d\lambda \circ (V, g)^{-1},
\]

since \( h \circ (U, g) \) is bounded and uniformly continuous on \( T_i \) for all \( 1 \leq i \leq k \). Clearly, we also have \( \text{supp}(\nu_n) \rightarrow \text{supp}(\lambda) \). Hence, analogously as in the proof of Theorem 2, we can show that \( B^{2/2}_{\tau_n} \subseteq B^{\varepsilon}_{\tau_n} \), where \( \tau = \lambda \circ (V, g)^{-1} \), 
\( \tau_n = \nu_n \circ (V_n, g_n)^{-1} \) and \( \varepsilon_n = \varepsilon/2 + 2\alpha_n + \gamma_n \). Since \( \varepsilon_n \rightarrow \varepsilon/2 \), there is \( N \) such that \( \varepsilon_n < \varepsilon \) for all \( n \geq N \). Hence, for all \( n \geq N \), we have

\[
\tau_n(B^{2}_{\tau_n}) \geq \tau_n(B^{\varepsilon}_{\tau_n}) \geq \tau_n(B^{2/2}_{\tau}) = \\
\nu_n(\{t : (U, g)(t) \in B^{2}_{\tau}\}) \geq \nu_n(\{g = f\}) \geq 1 - \varepsilon/2 > 1 - \varepsilon.
\]

Hence, \( g_n \) is an \( \varepsilon \)-equilibrium of \( G_n \). Since \( f \) takes values on \( P \) and \( \nu_n(\{g \neq f\}) < \varepsilon \), it follows that all but a fraction of players of \( G_n \) smaller than \( \varepsilon \) plays a pure strategy in \( g_n \).

**Proof of Theorem 4.** We may assume that \( U(t)(f(t), \lambda \circ f^{-1}) > U(t)(a, \lambda \circ f^{-1}) \) for all \( a \neq f(t) \) and \( t \in [0, 1] \). Let \( \gamma(t) = U(t)(f(t), \lambda \circ f^{-1}) - \max_{a \neq f(t)} U(t)(a, \lambda \circ f^{-1}) > 0 \). Also, we denote \( h = (U, f) \) and \( \tau = \lambda \circ h^{-1} \).

Since \( \mathcal{U} \) is separable, we can write \( \mathcal{U} = \bigcup_{i=1}^{\infty} A_{n,i} \) where \( \{A_{n,i}\} \) is a disjoint collection, each of its member being a Borel set with a diameter no greater than \( 1/n \). Denote by \( \mathcal{P} \) the collection of those set from

\[
\mathcal{N} = \{A_{n,1} \times \{a\}, \ldots, A_{n,n-1} \times \{a\}, \cup_{i=1}^{\infty} A_{n,i} \times \{a\}\}_{a \in A}
\]

that have strictly positive measure.

Let \( n \in \mathbb{N} \) be fixed. Let \( K_n \in \mathbb{N} \) be such that \( (K_n + 1)2^{-K_n} < 1/n \), and \( \eta_n = \min\{\tau(P)\}_{P \in \mathcal{P}_n} \). Let \( C_n \) be a compact subset of \([0, 1]\) satisfy \( \lambda([0,1] \setminus C_n) < \min\{\eta_n, 2^{-K_n}\} \), \( \{U(t)(a, \cdot)\}_t \) is equicontinuous at \( \tau_a \) and
Let $\gamma = \min_{t \in C_n} \gamma(t) > 0$ and let $\delta > 0$ be such that $\rho(\nu^t, \nu^A) < \delta$ implies that $|U(t)(a, \nu^A) - U(t)(a, \nu^A)| < \gamma$ for all $t \in C_n$.

Let $I_n \in \mathbb{N}$ be such that $\sum_{i=I_n}^{\infty} \tau_t(A_n, i) < 1/n$. For each $a \in A$, let $\{p_{n,i,a}\}_{i=1}^{I_n} \subset \mathbb{Q}$ be such that

$$|\hat{p}_{n,i,a} - \tau(A_n, i \times \{a\})| < \delta/(2I_n), i = 1, \ldots, I_n - 1,$$

and

$$\sum_{a \in A} \sum_{i=1}^{I_n} \hat{p}_{n,i,a} = 1.$$  \hfill (29)

Also, if $\tau(A_n, i \times \{a\}) = 0$, $i = 1, \ldots, I_n - 1$, let $\hat{p}_{n,i,a} = 0$, and if $\sum_{i=I_n}^{\infty} \tau(A_n, i \times \{a\}) = 0$ let $\hat{p}_{n,I_n} = 0$.

It follows that $|\hat{p}_{n,I_n} - \sum_{i=I_n}^{\infty} \mu(A_n, i)| < \delta/2$ Furthermore, there exists $N_n \in \mathbb{N}$, and $\{p_{n,i}\}_{i=1}^{I_n} \subset \mathbb{N}$ such that $\hat{p}_{n,i} = p_{n,i}/N_n$, $i = 1, \ldots, I_n$.

We define $\nu_n$ and $T_n$ similarly as in the proof of Lemma 5; however, we impose that $\text{supp}(\nu_n) \subset C_n$. This requirement can always be satisfied since $\lambda([0,1] \setminus C_n) < \min\{\tau(P)\}_P \in \mathcal{P}_n = \min\{\lambda(h^{-1}(P))\}_P \in \mathcal{P}_n$ implies that $\lambda(C_n \cap h^{-1}(P)) > 0$ for all $P \in \mathcal{P}$. Also, since $\lambda([0,1] \setminus C_n) < 2K_n$, we can define $\hat{p}_{n,k}$ as in the proof of Lemma 5 in a way that $\hat{p}_{n,k} \in C_n$ for all $k \leq K_n$, and $l = 1, \ldots, L_n, K_n$. We then define $T_n = \{t_{n,K_n+1}^{l} \}_{l=1}^{L_n, K_n}$ and $\nu_n$ to be the uniform distribution on $T_n$.

Note that $\nu_n \circ f_{T_n}^{-1}(\{a\}) = \sum_{i=1}^{I_n} \hat{p}_{n,i,a}$ and so

$$\left| \nu_n \circ f_{T_n}^{-1}(\{a\}) - \lambda \circ f^{-1}(\{a\}) \right| =$$

$$\sum_{i=1}^{I_n} \left(\hat{p}_{n,i,a} - \tau_A(A_n, i \times \{a\}) + \hat{p}_{n,I_n,a} - \tau_A(\cup_{i=I_n}^{\infty} A_n, i \times \{a\})\right)$$

$$< \sum_{i=1}^{I_n} \frac{\delta}{2I_n} < \frac{\delta}{2} < \delta.$$ \hfill (30)

This implies that $\rho(\nu_n \circ f_{T_n}^{-1}, \lambda \circ f^{-1}) < \delta$. Since $\text{supp}(\nu_n) \subset C_n$, it follows that $f_{T_n}$ is an equilibrium of $G_n = ((T_n, \nu_n), U|_{T_n})$.

Since

$$M^{n,K_n+1} < \frac{M_{K_n+1}}{2K_n} + \frac{K_n}{2K_n} < \frac{K_n + 1}{2K_n} < \frac{1}{n},$$ \hfill (31)

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it follows that \( \text{supp}(\nu_n) \to \text{supp}(\lambda) \). Also, by the same argument as in the proof of Lemma 5, we see that \( \nu_n \circ (U|_{T_n}, f|_{T_n})^{-1} \Rightarrow \lambda \circ (U, f)^{-1} \). This completes the proof. ■

**Proof of Theorem 5.** Analogous to the proof of the sufficiency part of Theorem 1. ■

**Proof of Theorem 6.** Let \( \{\nu_n\} \) and \( \{T_n\} \) be such that \( \nu_n \) is uniform on \( T_n \), a finite subset of \([0, 1] \), \( T_n \to [0, 1] \) and \( \nu_n \circ U|_{T_n}^{-1} \Rightarrow \lambda \circ U^{-1} \) (existence is guaranteed by Lemma 5).

Since \( A \) is a compact, convex subset of metric vector space, \( U(t) \) is continuous and \( a \mapsto U(t)(a, \tau) \) is quasi-concave for all \( t \in [0, 1] \) and all \( \tau \in \mathcal{M} \), the game \( G_n = ((T_n, \nu_n), U|_{T_n}, A) \) has an equilibrium \( f_n \) (see Reny [28]).

Define \( \tau_n = \nu_n \circ (U|_{T_n}, f_n)^{-1} \). Since \( \tau_{U,n} \Rightarrow \lambda \circ U^{-1} \) and \( \tau_{U,n} \) is tight for all \( n \) since \( U \) is a separable complete metric space, it follows that the family \( \{\tau_{U,n}\} \) is tight (see Hildenbrand [17, Proposition 32, p. 49]). Clearly, the family \( \{\tau_{\mathcal{M},n}\} \) is tight since \( \mathcal{M} \) is compact. Hence, by Proposition 35 in Hildenbrand [17], the family \( \{\tau_n\} \) is tight and so has a convergent subsequence \( \tau \) (Hildenbrand [17, Proposition 31, p. 49]). Clearly, \( \tau_U = \lambda \circ U^{-1} \), and \( \tau \) is an limit equilibrium distribution of \( \lambda \circ U^{-1} \). ■

**Proof of Theorem 7.** It follows from Lemma 6. ■

**References**


