Strategy-Proofness and Efficiency in a Mixed-Good Economy with Single-Peaked Preferences

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Abstract

We conduct an analysis of the implications of strategy-proofness and efficiency in a model of a mixed-good economy.

Following Côrte-Real (2001), we decompose the mixed good into two components - a private good and a public good - and restrict the domain to separably single-peaked and lexicographic preferences, where the private good is the primary concern of any agent. A rule simultaneously chooses a level of the public good that is also the total amount available for the private good, and its distribution among the agents. We then study the implications of strategy-proofness and efficiency in this model. Adding the requirements of continuity and symmetry, we achieve a fundamental result: that any rule satisfying this set of properties must allocate its chosen public good level according to a uniform distribution. We then extract all the additional implications of this set of properties, namely that the rule must have a closed range, and conclude that they are less demanding on how the rule chooses the public good level, allowing for but not requiring the use of the uniform generalized median-voter rule.

Both strategic and normative concerns lead us however to a clear recommendation: the use of a uniform distribution, regardless of the associated choice for the public good level.

Keywords: Mixed goods, single-peaked, strategy-proof, uniform distribution.  
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1 Introduction

The social choice literature has devoted its attention to the study of properties of allocation rules in both models of single-peaked preferences over a private good and models of single-peaked preferences over a public good. In Côrte-Real (2001), we introduced another model of an economy with a mixed good where agents also have single-peaked preferences.

Mixed goods are those that combine a component with the characteristics of a private good with another component that is nondepletable and indivisible and therefore analogous to a public good. Education is an obvious example - each individual draws a private benefit from his education level that enables him to acquire income in the future, but society also benefits from the interaction with a more skilled member. Similarly, health can be considered a mixed good.

In our analysis, we consider an economy with limited resources. These resources are jointly owned by all the agents in the economy and they can be used for either a mixed good or an alternative public good such as defense. We decompose the mixed good into two: a 'private good' (such as an agent’s education level expressed in terms of the amount of resources that is used for that agent’s education) and a ‘public good’ (such as the total amount of a country’s resources applied to education) where the level of the ‘public good’ simply becomes the total amount of the ‘private good’ that is distributed among the agents. From here on, in a slight abuse of vocabulary, we use the terms "private good" to refer to the private part of the mixed good and "public good" to refer to the public part of the mixed good.

The notion of single-peaked preferences in this setting is more complex: each person has single-peaked preferences over both the amount of the good he is actually awarded and about the whole amount to be distributed among all agents. Each person desires a specific level of the private good, which we name his "private peak", and getting either more or less than that amount makes him worse off\footnote{Throughout the paper, we use the terms "better off" and "worse off" to denote strict preference relations. Similarly, the terms "increase", "decrease", "greater" and "smaller" are used in the strict sense.}.

\footnote{We therefore assume that preferences with respect to this private good can be satiated. Although the possibility of satiation is empirically frequent, the standard economic assumption on preferences is monotonicity; the apparent contradiction can easily be solved by adding relevant constraints to models with monotonic preferences that will then yield reduced models with single-peaked preferences. In this case, we could start with a model where more education is better for each agent, adding also a time constraint for each individual and another time-consuming good such as leisure. This model would then yield a reduced model focusing only on education where each agent would desire a specific level of education and the further away from that point in either direction, the worse off the agent would be - i.e. the agent’s preference over his education level would be single-peaked.} Similarly, since the resources used for the total
provision of the private good (i.e. for the ‘public good’) have an alternative use, each person also desires a specific level of the available resources to be used for this public good - we name that level his ”public peak” - and using either more or less of the country’s resources for that good would make him worse off. So each person has a single-peaked preference over the public good as well.

We introduce two simplifying assumptions on preferences:

- **separable preferences**: each agent’s preferences over the private good are the same regardless of the level of public good and, similarly, preferences over the public good are independent of the level of private good an agent gets; this is a very useful assumption that allows us to analyze private-side preferences and public-side preferences separately;

- **lexicographic preferences**: we also restrict the domain and work with single-peaked lexicographic preferences - each agent does care about the public good, but he cares first about the amount of the good he gets for himself.

Having the private component of a mixed good be relatively more important than its public component seems to be a good behavioral assumption: using the example of health, a person cares first and foremost about the amount of health resources available to him, that has more immediate consequences, and then about the amount available to the whole country. The preferences for the public component of the mixed good seem to be a secondary concern. Still, although qualitatively reasonable, the very definite ’ordering’ of the preferences between private and public good that is imposed by lexicographic preferences is somewhat extreme from a quantitative viewpoint and gives rise to possible continuity issues. The results therefore require a careful interpretation.

However, besides allowing for a tractable model, the restriction to lexicographic preferences is probably the smallest step one could take going from the private good case to the mixed good case - and this enables us to test the robustness of the results that have been established for the private good case by checking the degree to which they can be extended to this new setting.

In Côrte-Real (2001), we conducted a normative analysis of this model and found different characterizations of allocation rules according to different sets of desirable properties, that included efficiency\(^3\) and no-envy\(^4\). Since these rules are not likely to achieve these desirable properties if they can be

\(^3\)A feasible allocation is efficient if there is no other feasible allocation that all agents like at least as much and at least one agent prefers.

\(^4\)An allocation is envy-free if no agent prefers another agent’s bundle to his own.
manipulated to an agent’s advantage, we now intend to conduct a strategic analysis of the model and determine whether the rules are strategy-proof i.e. whether they give an agent the incentives to reveal his true preferences in the associated direct revelation game, regardless of whether the other agents are revealing theirs.

Strategy-proofness ensures implementability in dominant strategies, but it is a very strong requirement. Nevertheless, the domain restriction allows us to overcome the negative Gibbard-Satterthwaite result and to achieve strategy-proofness and efficiency with a non-dictatorial rule.

A similar positive result was also achieved in the strategic analysis of the private-good model that established the relevance of the uniform rule. Sprumont (1991) proved that the uniform rule is the only anonymous rule that is simultaneously efficient and strategy-proof and Ching (1994) achieved the same result by dropping anonymity and imposing the weaker requirement of symmetry (or equal treatment of equals). In private-good models with single-peaked preferences, there is a fixed amount $\Omega$ of the good to be divided among the agents, no free disposal, and for each agent $i$ in the set $N$ there is an optimal level of the good, his peak $p(R_i)$, and getting an amount either greater or smaller than his peak makes him worse off. If the available resources are not enough to give every agent his peak, the uniform rule starts distributing the available resources equally among the agents until the agent with the lowest peak reaches that peak (at that point, all the other agents are getting that amount as well); it then leaves that agent at his peak and goes on distributing the remaining amount equally among the remaining agents, until the agent with the second lowest peak reaches that peak, and this sequential process of equal division continues until the resources are exhausted (i.e. the agents with the highest peaks all receive the same amount at the end and all the agents whose peaks are below that amount receive their peaks). More formally, if $\Omega \leq \sum_{i \in N} p(R_i)$, the uniform rule is defined as $x_i = \min \{p(R_i), \lambda\}$ for all $i \in N$, where $\lambda$ is determined by feasibility. If the available resources exceed the sum of the peaks, the uniform rule works as in the previous case until all agents receive their peaks and then starts allocating the remaining available resources entirely to the agent with the lowest peak until he reaches the second lowest peak; it then goes on distributing the remaining amount equally among the two agents with the two lowest peaks until they reach the third lowest peak.

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5Thomson (1999) devotes a chapter to models of single-peaked preferences over a private good and conducts an extensive study of the properties of the uniform rule, also presenting several characterizations of the rule based on axioms that are related to notions of equity and solidarity.

6A rule is anonymous if it does not depend on the names of the agents.

7In our model, however, the assumption of free disposal holds.
and this process continues until the resources are exhausted (i.e. the agents with the smallest peaks get the same amount at the end and the agents whose peaks are above that amount receive their peaks). More formally, if \( \Omega \geq \sum_{i \in N} p(R_i) \), the uniform rule is defined as \( x_i = \max \{p(R_i), \lambda\} \) for all \( i \in N \), where \( \lambda \) is determined by feasibility. For simplicity, we have omitted from the explanation the case where two or more agents have the same private peak but the underlying logic is the same. The uniform rule basically works as if it defined a set analogous to the Walrasian budget set \([0, \lambda]\) in the first case and \([\lambda, \Omega]\) in the second case) and asked agents to select their optimal point from that set. As a consequence of the normative requirement of no-envy, our previous axiomatic analysis of the mixed-good model focused on rules that allocate the chosen level of public good (i.e. the total available amount of private good) among the agents according to a uniform distribution, which is simply defined like the uniform rule for any given public good level. As we conduct a strategic analysis of the model and focus on informational and incentive issues, the need for the use of a uniform distribution is reinforced.

Another related model in which there exists a non-dictatorial rule that combines strategy-proofness and efficiency is the model of single-peaked preferences over a public good. In public good models with single-peaked preferences, a public good level must be chosen out of a maximal available amount \( \Omega \); for each agent there is an optimal level of the public good, his peak \( p(R_i) \), and an amount either greater or smaller than his peak makes him worse off. Moulin (1980) characterized the generalized median-voter rule (or generalized Condorcet-winner rule), that takes the \( N \) peaks and up to \( N - 1 \) constants and chooses the median of these points, as the unique peak-only\(^8\) rule satisfying strategy-proofness, efficiency and anonymity. Ching (1997) established another characterization replacing peak-only with continuity\(^9\). The analog of this rule for the mixed-good model also plays a relevant role in our analysis.

Unlike some models with more than one public good studied by Border and Jordan (1983), Zhou (1991) and Barberà and Jackson (1994), our mixed-good setting is such that we have no conflict between strategy-proofness and efficiency in a multidimensional model - and these properties are actually compatible with additional requirements. Our first result shows that a uniform distribution of the public good level must be used in any strategy-proof, continuous and efficient rule that also satisfies symmetry. We then proceed to extract all the implications of this set of properties and find a characterization of the set of rules that satisfy them. We conclude that even though

\(^8\)A rule is peak-only if it depends only on the peaks of the agents: changing the agents’ preferences but leaving the peaks unchanged does not alter the choice of the rule.

\(^9\)A rule is continuous if its choice varies continuously with the agents’ preferences.
we must have a specific distribution of the public good level, we have more flexibility in the choice of that level. Namely, although a generalized median-voter choice of the public good level coupled with its uniform distribution is an example of a rule that meets all the axioms, it is not the only such rule - and this is a consequence of the restriction to lexicographic preferences that attribute a secondary degree of relevance to the public good.

The paper is organized as follows: In section 2, we present a model that is based on these assumptions and describe the corresponding set of efficient allocations. In section 3, we focus on strategy-proofness, and present and prove the main results. In the last section, we summarize the main findings.

2 The model

A fixed social endowment $E \in \mathbb{R}_+$ of resources is available. A public good level $\Omega \leq E$ must be chosen and divided among the agents in the set $N$. Each agent $i \in N$ has a separably single-peaked lexicographic preference relation $R_i = (R^p_i, R^q_i)$ defined over $[0, E] \times [0, E]$\footnote{Given that, unlike Côrte-Real (2001), we will not be dealing with changes in $E$, only preferences over the restricted domain $[0, E] \times [0, E]$ will be relevant. Rather than defining preferences over the original domain $\mathbb{R}_+ \times \mathbb{R}_+$, we therefore introduce this simplification.}. We let $R_i$ and $P_i$ denote respectively weak and strict preference, and $I_i$ denote indifference.

Letting $p(R^p_i)$ be agent $i$’s ’private good’ peak amount and $p(R^q_i)$ be agent $i$’s ’public good’ peak amount, we have that:

- for all $x_i$ and $x_i'$, if $x'_i < x_i \leq p(R^p_i)$ or $x'_i > x_i \geq p(R^q_i)$, then $x_i P^p_i x'_i$;
- for all $\Omega$ and $\Omega'$, if $\Omega' < \Omega \leq p(R^q_i)$ or $\Omega > \Omega' \geq p(R^q_i)$, then $\Omega P^q_i \Omega'$;
- for all $(x_i, \Omega)$ and $(x'_i, \Omega')$, $(x_i, \Omega) P_i(x'_i, \Omega')$ if either $x_i P^p_i x'_i$, or $x_i I^q_i x'_i$ and $\Omega P^q_i \Omega'$.

Let $\mathcal{R}^{sp,lex}$ denote the class of all such preference relations. An economy is defined by the preference profile $R = (R_i)_{i \in N} \in \mathcal{R}^{N,lex}_i$\footnote{We could include $E$ in the definition of an economy, as we did in Côrte-Real (2001). However, given that throughout the analysis we will focus on axioms that deal only with changes in preferences while keeping $E$ fixed, we chose to denote an economy solely by its preference profile in order to simplify the notation. The analysis can easily be extended to the more general definition.}.

A feasible allocation is a pair $(\Omega, x)$ where $\Omega \leq E$ and the list $x = (x_i)_{i \in N} \in \mathcal{R}^{N}$ is such that $\sum_{i \in N} x_i = \Omega$. A single-valued allocation rule associates each preference profile with a unique allocation and must therefore determine both a public good level $\Omega$ - that is also the total amount available for the private good - and the distribution of that $\Omega$ among the agents i.e.
a list \((x_i)_{i \in N}\) where \(x_i\) is agent \(i\)'s private good amount, and the sum of all \(x_i\) equals \(\Omega\).

A feasible allocation is (Pareto-)efficient if there is no other feasible allocation that all agents like at least as much and at least one agent prefers.

**Definition 1** A rule is efficient if for all \(R = (R_i)_{i \in N} \in \mathbb{R}^N_{\text{sp,lex}},\) and all feasible \((\Omega, x)\) chosen by the rule for \(R\), there is no feasible \((\Omega', x')\) such that for all \(i \in N, (x_i', \Omega') R_i (x_i, \Omega)\) and for some \(i \in N, (x_i', \Omega') P_i (x_i, \Omega)\).

As we established in Côrte-Real (2001), given the social endowment \(E \in \mathbb{R}^+_+\), a feasible allocation \((\Omega, x)\) for an economy defined by the preference profile \(R = (R_i)_{i \in N} \in \mathbb{R}^N_{\text{sp,lex}}\) is efficient if and only if:

- whenever the social endowment is not greater than the sum of the private peaks, either the public good level is the social endowment and no agent gets more than his private peak, or it is less than the social endowment, no agent gets more than his private peak and at least one agent gets his private peak and has a public peak below or equal to that level of public good; i.e. for \(E \leq \sum_{i \in N} p(R_i^e),\)

  - either \(\Omega = E\) and \(x_i \leq p(R_i^e)\) for all \(i \in N\)
  
  - or \(\Omega < E, x_i \leq p(R_i^e)\) for all \(i \in N\), and there exists an agent \(j \in N\) such that \(x_j = p(R_j^e)\) and \(p(R_j^0) \leq \Omega;\)

- whenever the social endowment is greater than the sum of the private peaks, either the public good level is the sum of the private peaks and each agent gets his private peak; or it is less than the sum of the private peaks, no agent gets more than his private peak and at least one agent gets his private peak and has a public peak below or equal to that level of public good; or it is more than the sum of the private peaks, no agent gets less than his private peak and at least one agent gets his private peak and has a public peak above or equal to that level of public good;

  i.e. for \(E > \sum_{i \in N} p(R_i^e),\)

  - either \(\Omega = \sum_{i \in N} p(R_i^e)\) and \(x_i = p(R_i^e)\) for all \(i \in N,\)
  
  - or \(\Omega < \sum_{i \in N} p(R_i^e), x_i \leq p(R_i^e)\) for all \(i \in N,\) and there exists an agent \(j \in N\) such that \(x_j = p(R_j^e)\) and \(p(R_j^0) \leq \Omega,\)
  
  - or \(E \geq \Omega > \sum_{i \in N} p(R_i^e), x_i \geq p(R_i^e)\) for all \(i \in N,\) and there exists an agent \(j \in N\) such that \(x_j = p(R_j^e)\) and \(p(R_j^0) \geq \Omega.\)
3 Strategy-proofness

We are interested in single-valued rules that provide the agents with the incentives to truthfully reveal their preferences in the associated direct revelation game, i.e. we are interested in incentive-compatible rules. The strongest incentive compatibility condition is implementability in dominant strategies or strategy-proofness, which requires that no agent benefit from misrepresenting his preferences regardless of what the other agents do.

**Definition 2** A rule is strategy-proof if for all \( R \in \mathbb{R}^{N}_{sp,lex} \), for all \( i \in N \), for all \( R_i \in \mathbb{R}_{sp,lex} \), if \((\Omega, x)\) is selected by the rule for \( R \) and \((\Omega', x')\) is selected by the rule for \( R' = (R_i, R_{-i}) \), then \((\Omega, x) \rightarrow (\Omega', x')\).

We would like to analyze the extent to which strategy-proofness narrows down the set of efficient rules in this model. Our main results are however also based on an additional axiom - continuity - that requires that the rule vary continuously with preferences. The imposition of continuity intends to ensure that the rules are 'well-behaved' under a 'trembling-hand' i.e. in the event of small errors in assessing a preference relation, we would like to have no jumps or sudden variations in the choices each rule makes. Further ahead, we will see an example of the type of rule we are excluding by adding this requirement.

Formally, for any two preference relations \( R_i \) and \( R'_i \) defined on \([0, E] \times [0, E]\), let \( r_i^x : [0, E] \rightarrow [0, E] \) be the function that defines the private-side preferences using the private-side indifference relations: for all \( x_i \in [0, p(R_i^x)] \), \( r_i^x(x_i) = y_i \) if there is a \( y_i \in [p(R_i^x), E] \) such that \( y_i I_i^x x_i \) and \( r_i^x(x_i) = E \) otherwise; similarly, for all \( x_i \in [p(R_i^x), E] \), \( r_i^x(x_i) = y_i \) if there is a \( y_i \in [0, p(R_i^x)] \) such that \( y_i I_i^x x_i \) and \( r_i^x(x_i) = 0 \) otherwise; and let \( r_i^\Omega : [0, E] \rightarrow [0, E] \) be the function that defines the public-side preferences using the public-side indifference relations: for all \( \Omega \in [0, p(R_i^\Omega)] \), \( r_i^\Omega(\Omega) = \Omega' \) if there is a \( \Omega' \in [p(R_i^\Omega), E] \) such that \( \Omega' I_i^\Omega \Omega \) and \( r_i^\Omega(\Omega) = E \) otherwise; similarly, for all \( \Omega \in [p(R_i^\Omega), E] \), \( r_i^\Omega(\Omega) = \Omega' \) if there is a \( \Omega' \in [0, p(R_i^\Omega)] \) such that \( \Omega' I_i^\Omega \Omega \) and \( r_i^\Omega(\Omega) = 0 \) otherwise. Now let the distance between the private-side preferences be \( d^x(R_i^x, R'_i) = \max \{|r_i^x(x_i) - r_i'^x(x_i)| : x_i \in [0, E]\} \) and the distance between the public-side preferences be \( d^\Omega(R_i^\Omega, R'_i) = \max \{|r_i^\Omega(\Omega) - r_i'^\Omega(\Omega)| : \Omega \in [0, E]\} \). We define the distance between the two preference relations as \( d(R_i, R'_i) = \max \{d^x(R_i^x, R'_i), d^\Omega(R_i^\Omega, R'_i)\} \). A sequence of preference relations \( \{R_i^n\} \in \mathbb{R}_{sp,lex} \) converges to \( R_i \), denoted by \( R_i^n \rightarrow R_i \), if as \( n \rightarrow \infty \), \( d(R_i, R_i^n) \rightarrow 0 \).

**Definition 3** A rule is continuous if for all \( R_i \in \mathbb{R}^{N}_{sp,lex} \), for all \( i \in N \), for all \( \{R_i^n\} \in \mathbb{R}_{sp,lex} \), where \((\Omega, x)\) is selected by the rule for \( R \) and \((\Omega^n, x^n)\) is selected by the rule for \( R^n = (R_i^n, R_{-i}) \), if \( R_i^n \rightarrow R_i \), then \((\Omega^n, x^n) \rightarrow (\Omega, x)\).
We now define two invariance properties that are useful for our analysis. The first is a weakening of the peak-only requirement that we used in Côrte-Real (2001) and that parallels the axiom that Moulin(1980) used in the public-good model: a rule is peak-only if its choice is not altered when agents’ preferences change but their peaks remain the same. Although we introduced the peak-only requirement in Côrte-Real (2001) to ensure the informational simplicity of a rule, the strategic relevance of this axiom (as well as of its weaker version that we present next) is intuitive: the less a rule depends on the information on preferences, the harder it will be for agents to manipulate that rule. Formally, we have that:

**Definition 4** A rule is **peak-only** if, for all \( R, R' \in \mathcal{R}_{sp,lex}^N \) such that, for all \( i \in N \), \( p(R_x^i) = p(R'^i_x) \) and \( p(R^\Omega_i) = p(R'^\Omega_i) \), it chooses the same \((\Omega, x)\) for \( R \) and \( R' \).

The weaker axiom that we will focus on states that if one agent’s preferences change without changing his peaks, and all the other agents’ preferences remain the same, what that agent gets does not change:

**Definition 5** A rule is **weakly peak-only** if, for all \( i \in N \), for all \( R, R' \in \mathcal{R}_{sp,lex}^N \) such that \( R_{-i} = R'_{-i} \), \( p(R_x^i) = p(R'^i_x) \) and \( p(R^\Omega_i) = p(R'^\Omega_i) \), it chooses the same \( \Omega \) and the same \( x_i \) for \( R \) and \( R' \).

Note that in this case we do not require the rule to choose the same distribution of \( \Omega \) as an agent’s preferences change but we do require that what that agent receives of the private good remains the same.

The other invariance property requires that the rule be uncompromising. Border and Jordan (1983) introduced a similar notion for the public-good model; in that model, an uncompromising rule does not alter the choice of the public good level if an agent whose peak was lower than that choice announces a different peak that is still not greater than the original choice. The underlying idea of an uncompromising rule in this model is the same, namely that the rule is not influenced by extreme positions that the agents might take and therefore by any compromise between his actual position and an extreme position; however, the definition is more complex in this setting that combines private-side with public-side preferences in a lexicographic manner. Again, the strategic relevance of this property is intuitive: an agent will not have an incentive to announce a more extreme position since that alternative announcement will have no bearing on the choice of the rule.

**Definition 6** A rule is **uncompromising** if for all \( R \in \mathcal{R}_{sp,lex}^N \), for all \( i \in N \), for all \( R'_i \in \mathcal{R}_{sp,lex}^N \), if the rule chooses \((\Omega, x)\) for \( R \) and \((\Omega', x')\) for \((R'_i, R_{-i})\), then all of the following hold:
• if $R_i^x = R_i^{x'}$, then $x'_i = x_i$ i.e. the rule is **public-side neutral for the private component** of agent $i$’s bundle;

• if $R_i^x = R_i^{x''}$ and $(\Omega < p(R_i^x) \land \Omega \leq p(R_i^{x''}))$ or $(\Omega > p(R_i^x) \land \Omega \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **public-side uncompromising**;

• if $R_i^x = R_i^{x''}$ and $(x_i < p(R_i^x) \land x_i \leq p(R_i^{x''}))$ or $(x_i > p(R_i^x) \land x_i \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **private-side uncompromising**;

As a consequence of the last two cases, we also have that if $(\Omega < p(R_i^x) \land \Omega \leq p(R_i^{x''}))$ or $(\Omega > p(R_i^x) \land \Omega \geq p(R_i^{x''}))$ and $(x_i < p(R_i^x) \land x_i \leq p(R_i^{x''}))$ or $(x_i > p(R_i^x) \land x_i \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **both-sides uncompromising**.

For our next proposition, however, we concentrate on a weaker version of the uncompromising property that focuses only on the invariance of agent $i$’s bundle as his preferences change:

**Definition 7** A rule is **weakly uncompromising** if for all $R \in \mathbb{R}^N_{\text{sp,lex}}$, for all $i \in N$, for all $R_i' \in \mathbb{R}_{\text{sp,lex}}$, if the rule chooses $(\Omega, x)$ for $R$ and $(\Omega', x')$ for $(R_i', R_{-i})$, then all of the following hold:

• if $R_i^x = R_i^{x'}$, then $x'_i = x_i$ i.e. the rule is **public-side neutral for the private component** of agent $i$’s bundle;

• if $R_i^x = R_i^{x''}$ and $(\Omega < p(R_i^x) \land \Omega \leq p(R_i^{x''}))$ or $(\Omega > p(R_i^x) \land \Omega \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **weakly public-side uncompromising for the public component** of agent $i$’s bundle;

• if $R_i^x = R_i^{x''}$ and $(x_i < p(R_i^x) \land x_i \leq p(R_i^{x''}))$ or $(x_i > p(R_i^x) \land x_i \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **weakly private-side uncompromising for both the private and the public components** of agent $i$’s bundle;

As a consequence of the last two cases, if $(\Omega < p(R_i^x) \land \Omega \leq p(R_i^{x''}))$ or $(\Omega > p(R_i^x) \land \Omega \geq p(R_i^{x''}))$ and $(x_i < p(R_i^x) \land x_i \leq p(R_i^{x''}))$ or $(x_i > p(R_i^x) \land x_i \geq p(R_i^{x''}))$, then $x'_i = x_i$ and $\Omega' = \Omega$ i.e. the rule is also **weakly both-sides uncompromising for both the private and the public components** of agent $i$’s bundle.

Finally, and before stating the next proposition, we need to introduce one more axiom based on a notion introduced by Barberà and Jackson (1994) and that is also closely related to strategy-proofness: a rule satisfies closed range if, keeping the other agents’ preferences fixed, both the private amount
an agent receives and the public good level can only vary within a closed interval as his preferences change; if the agent’s peak is in the interval, he gets his peak; otherwise, he gets the limit of the interval that is closest to his peak. Once more, the strategic relevance of this axiom is very intuitive: if the rule has closed range, an agent will get his preferred point out of the fixed interval and will therefore have no incentive to announce different peaks.

**Definition 8** A rule has a closed range if for all \( i \in \mathbb{N} \), for all \( R \in \mathbb{R}_{sp,lex}^{\mathbb{N}} \),

- if \( p(R^x_i) = 0 \), the rule chooses \( x_i \) for \( R \); if \( p(R^x_i) = E \), the rule chooses \( x_i \geq x_j \) for \( R \); and for any \( p(R^x_i) \), the rule chooses \( x_i = \text{med}\{p(R^x_i), x_i, x_j\} \) for \( R \).

- for all \( x_i \in [x_j, x_i] \), if \( p(R^x_i) \) is such that the rule chooses \( x_i \) for \((R_i, R_{-i})\), if \( p(R^{\Omega}_i) = 0 \), the rule chooses \( \Omega(x_i) \) for \( R \); if \( p(R^{\Omega}_i) = E \), the rule chooses \( \Omega(x_i) \geq \Omega(x_i) \) for \( R \); and for any \( p(R^{\Omega}_i) \), the rule chooses \( \Omega = \text{med}\{p(R^{\Omega}_i), \Omega(x_i), \Omega(x_i)\} \) for \( R \).

Proposition 1 establishes the logical relations among all these axioms. It states that imposing strategy-proofness and continuity restricts the set of rules to the ones that have a closed range, are weakly peak-only and weakly uncompromising\(^{12}\). We also find that a rule has a closed range if and only if it is strategy-proof and either weakly peak-only or weakly uncompromising - since the last two properties are equivalent for a strategy-proof rule.

**Proposition 1** If a rule is strategy-proof and continuous, it is also weakly peak-only, weakly uncompromising and has a closed range. Moreover, a rule has a closed range if and only if it is strategy-proof and weakly peak-only or, equivalently, if and only if it is strategy-proof and weakly uncompromising.

The proof of the Proposition is divided into several lemmas.

**Lemma 1** If a rule is strategy-proof and continuous, it is also weakly peak-only.

**Proof.** Assume not, towards a contradiction. Then, for some \( i \in \mathbb{N} \), and some \( R, R' \in \mathbb{R}_{sp,lex}^{\mathbb{N}} \) such that \( R_{-i} = R'_{-i} \), \( p(R^x_i) = p(R'^x_i) \) and \( p(R^{\Omega}_i) = p(R'^{\Omega}_i) \), the rule chooses \( \Omega \) and \( x_i \) for \( R \) and \( \Omega' \) and \( x'_i \) for \( R' \) where either \( \Omega \neq \Omega' \) or \( x_i \neq x'_i \).

\(^{12}\)Ching (1997) presents a similar result for the public-good model: that for a strategy-proof rule, being continuous, uncompromising, peak-only or having a closed range are all equivalent.
Without loss of generality, assume that $x_i \leq p(R_i^v)$. If $x_i \neq x_i'$, and for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i'$, either $x_i < p(R_i^v)$ and $x_i' < x_i$ or $x_i' \geq r_i^v(x_i)$, or $x_i = p(R_i^v)$ and $x_i' \neq p(R_i^v)$.

- if $x_i' < x_i \leq p(R_i^v) = p(R_i^{v''})$, an agent with preferences $R_i'$ would benefit from announcing $R_i$ and the rule would not be strategy-proof;

- if $x_i < p(R_i^v)$ and $x_i' \geq r_i^v(x_i)$, then we can find a sequence $R_i^{v''} \rightarrow R_i'$ where $v \in [0,1] \cap Q$ and $r_i^{v''} = (1-v).r_i^v + v.r_i^{v''}$ and $r_i^{v''} = (1-v).r_i^v + v.r_i^{v''}$ which means that $p(R_i^{v''}) = p(R_i^v)$ and $p(R_i^{v''}) = p(R_i^{v''})$ and that $R_i^{v''} = R_i$ and $R_i^{v''} = R_i'$. For continuity, we can find a $v$ and the corresponding $R_i^{v''}$ such that $x_i^{v''} = p(R_i^{v''}) = p(R_i^v)$. But then $R_i$ would benefit from announcing that $R_i^{v''}$ instead and the rule would not be strategy-proof.

- if $x_i = p(R_i^v) = p(R_i^{v''})$ and $x_i' \neq p(R_i^v)$, then an agent with preferences $R_i'$ would benefit from announcing $R_i$ and the rule would not be strategy-proof.

Therefore, $x_i = x_i'$.

Without loss of generality, assume that $\Omega \leq p(R_i^\Omega)$. Given that $x_i = x_i'$, if $\Omega \neq \Omega'$, and for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i'$, either $\Omega < p(R_i^\Omega)$ and $\Omega' < \Omega$ or $\Omega' \geq r_i^\Omega(\Omega)$, or $\Omega = p(R_i^\Omega)$ and $\Omega' \neq p(R_i^\Omega)$.

- if $\Omega' < \Omega \leq p(R_i^\Omega) = p(R_i^{\Omega'})$, an agent with preferences $R_i'$ would benefit from announcing $R_i$ and the rule would not be strategy-proof;

- if $\Omega < p(R_i^\Omega)$ and $\Omega' \geq r_i^\Omega(\Omega)$, then we can find a sequence $R_i^{v''} \rightarrow R_i'$ where $v \in [0,1] \cap Q$ and $r_i^{v''} = (1-v).r_i^v + v.r_i^{v''}$ and $r_i^{v''} = (1-v).r_i^v + v.r_i^{v''}$ which means that $p(R_i^{v''}) = p(R_i^v)$ and $p(R_i^{v''}) = p(R_i^{v''})$ and that $R_i^{v''} = R_i$ and $R_i^{v''} = R_i'$. For continuity, we can find a $v$ and the corresponding $R_i^{v''}$ such that $\Omega'' = p(R_i^{v''}) = p(R_i^\Omega)$. But then we already know that $x_i^{v''} = x_i = x_i'$ must hold throughout the sequence and $R_i$ would benefit from announcing that $R_i^{v''}$ instead and the rule would not be strategy-proof.

- if $\Omega = p(R_i^\Omega) = p(R_i^{\Omega'})$ and $\Omega' \neq p(R_i^\Omega)$, then an agent with preferences $R_i'$ would benefit from announcing $R_i$ and the rule would not be strategy-proof.

Therefore, $\Omega = \Omega'$.

Lemma 2 A strategy-proof rule is weakly peak-only if and only if it is weakly uncompromising.

Proof. Step 1: we prove that a strategy-proof rule that is weakly peak-only is weakly uncompromising. Assume not. Then, for some $R \in \mathcal{R}_{sp,lex}^N$, some
$i \in N$, and some $R_i' \in \mathcal{R}_{sp,lex}$, if the rule chooses $(\Omega, x)$ for $R$ and $(\Omega', x')$ for $(R_i', R_{-i})$, one of the following does not hold:

- if $R_i'' = R_i'''$, then $x_i' = x_i$ i.e. the rule is public-side neutral for the private component of agent $i$’s bundle. Assume this does not hold, and, without loss of generality, assume that $x_i \leq p(R_i'')$. Then, since $R_i'' = R_i'''$, either $x_i' = x_i$ or $x_i' = r_i''(x_i)$ for strategy-proofness, so that neither benefits from announcing the other agent’s preferences. If $x_i = p(R_i'')$, we then have $x_i' = p(R_i'') = x_i$. If $x_i < p(R_i'')$, assume towards a contradiction, that $x_i' = r_i''(x_i)$. Then we can find a preference relation $R_i''$ such that $p(R_i'') = p(R_i'')$, $p(R_i'''') = p(R_i'')$ and $r_i'''(x_i) > x_i' = r_i''(x_i) > p(R_i'')$. Since the rule is weakly peak-only, if agent $i$’s preference relation were $R_i''$ instead of $R_i$, the rule would still pick $x_i'' = x_i$ and $\Omega'' = \Omega$. But then an agent with preferences $R_i''$ would benefit from announcing $R_i$ instead and the rule would not be strategy-proof.

Therefore, $x_i = x_i'$.

- if $R_i'' = R_i'''$ and $(\Omega < p(R_i'')$ and $\Omega \leq p(R_i'')$) or $(\Omega > p(R_i'')$ and $\Omega \geq p(R_i'')$), then $x_i' = x_i$ and $\Omega' = \Omega$ i.e. the rule is also public-side uncompromising for the public component of agent $i$’s bundle. Assume this does not hold. We already know that $x_i' = x_i$ from the previous case. Without loss of generality, assume that $\Omega < p(R_i'')$. Given that $x_i = x_i'$, if $\Omega \neq \Omega'$, and for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i$, either $\Omega' < \Omega$ or $\Omega' \geq r_i''(\Omega)$.

- if $\Omega' < \Omega \leq p(R_i'')$, an agent with preferences $R_i$ would benefit from announcing $R_i$ and the rule would not be strategy-proof;

- if $\Omega < p(R_i'')$ and $\Omega' \geq r_i''(\Omega)$, then we can find a preference relation $R_i''$ such that $p(R_i'') = p(R_i'')$, $p(R_i'''') = p(R_i''')$ and $r_i''''(\Omega) > \Omega' \geq r_i''''(\Omega) > p(R_i'') = p(R_i''')$. Since the rule is weakly peak-only, if agent $i$’s preference relation were $R_i''$ instead of $R_i$, the rule would still pick $x_i'' = x_i$ and $\Omega'' = \Omega$. But then an agent with preferences $R_i''$ would benefit from announcing $R_i'$ instead and the rule would not be strategy-proof.

Therefore, $\Omega = \Omega'$.

- if $R_i'' = R_i'''$ and $(x_i < p(R_i'')$ and $x_i \leq p(R_i'')$) or $(x_i > p(R_i'')$ and $x_i \geq p(R_i'')$), then $x_i' = x_i$ and $\Omega' = \Omega$ i.e. the rule is also private-side uncompromising for both the private and the public components of agent $i$’s bundle. Assume this does not hold, and, without loss of generality, assume that $x_i < p(R_i'')$. If $x_i \neq x_i'$, and for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i$, either $x_i' < x_i$ or $x_i' \geq r_i''(x_i)$.

- if $x_i' < x_i \leq p(R_i'')$, an agent with preferences $R_i$ would benefit from announcing $R_i$ and the rule would not be strategy-proof;
-if $x_i < p(R_i^p)$ and $x'_i \geq r_i^p(x_i)$, then we can find a preference relation $R''_i$ such that $p(R''_i) = p(R_i^p)$, $p(R''_i) = p(R_i^{Q'})$ and $r''_i(x_i) > x'_i \geq r_i^p(x_i) > p(R_i^p) = p(R''_i)$. Since the rule is weakly peak-only, if agent $i$’s preference relation were $R''_i$ instead of $R_i$, the rule would still pick $x''_i = x_i$ and $\Omega'' = \Omega$. But then an agent with preferences $R''_i$ would benefit from announcing $R''_i$ instead and the rule would not be strategy-proof.

Therefore, $x_i = x'_i$. Assume now, without loss of generality, that $\Omega \leq p(R_i^{Q'})$. Given that $x_i = x'_i$, if $\Omega \neq \Omega'$, and for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i'$, either $\Omega < p(R_i^p)$ and $\Omega' < \Omega$ or $\Omega' \geq r_i^{Q'}(\Omega)$, or $\Omega = p(R_i^p)$ and $\Omega' \neq p(R_i^p)$.

-if $\Omega' < \Omega \leq p(R_i^{Q'})$, an agent with preferences $R_i'$ would benefit from announcing $R_i'$ and the rule would not be strategy-proof;

-if $\Omega < p(R_i^p)$ and $\Omega' \geq r_i^p(\Omega)$, then we can find a preference relation $R''_i$ such that $p(R''_i) = p(R_i^p)$, $p(R''_i) = p(R_i^{Q'})$ and $r''_i(\Omega) > \Omega' \geq r_i^p(\Omega) > p(R_i^p) = p(R''_i)$. Since the rule is weakly peak-only, if agent $i$’s preference relation were $R''_i$ instead of $R_i$, the rule would still pick $x''_i = x_i$ and $\Omega'' = \Omega$. But then an agent with preferences $R''_i$ would benefit from announcing $R''_i$ instead and the rule would not be strategy-proof.

-if $\Omega = p(R_i^p)$, $\Omega' \neq p(R_i^p)$, an agent with preferences $R_i'$ would benefit from announcing $R_i'$ and the rule would not be strategy-proof.

Therefore, $\Omega = \Omega'$.

Therefore, a strategy-proof rule that is weakly peak-only is weakly uncompromising.

Step 2: We now want to prove the reverse implication: that a strategy-proof rule that is weakly uncompromising must also be weakly peak-only. We need to check that, for all $i \in N$, and all $R, R' \in R_{sp, lex}^N$ such that $R_{-i} = R'_{-i}$, $p(R_i^p) = p(R_i^{Q'})$ and $p(R_i^p) = p(R_i^{Q'})$, the rule chooses $\Omega$ and $x_i$ for $R$ and $\Omega'$ and $x'_i$ for $R'$ where $\Omega = \Omega'$ and $x_i = x'_i$. Since the rule is weakly both-sides uncompromising, we only need to focus on the cases where agent $i$ is getting at least one of the peaks:

- if $x_i = p(R_i^p) = p(R_i^{Q'})$, let, without loss of generality, $\Omega < p(R_i^p) = p(R_i^{Q'})$; $x'_i = x_i$ for strategy-proofness, since otherwise the agent with preferences $R_i'$ would benefit from announcing $R_i$ instead. Also for strategy-proofness, so that an agent with preferences $R_i$ does not benefit from announcing $R_i'$, either $\Omega' = \Omega$ or $\Omega' \geq r_i^{Q'}(\Omega) > p(R_i^p) = p(R_i^{Q'})$. Then, since the rule is public-side neutral for the private component of agent $i$’s bundle and weakly public-side uncompromising for the public component of agent $i$’s
bundle, an agent with preferences \( R''_i \), where \( R''_i = p(R''_i) \) and \( R''_i \) is such that \( p(R''_i) = p(R''_i) \) and \( \Omega' > p(R''_i) \) then \( p(R''_i) = p(R''_i) \) would still receive \( x''_i = x'_i \) and \( \Omega'' = \Omega' \). But then an agent with preferences \( R''_i \) would benefit from announcing \( R_i \), and the rule would not be strategy-proof.

- if \( \Omega = p(R''_i) = p(R''_i) \), let, without loss of generality, \( x_i < p(R''_i) = p(R''_i) \); we can decompose the change in preferences from \( R_i \) to \( R_i \) into two steps: first from \( R_i \) to \( R_i \), where \( R''_i = R''_i \) and \( R''_i = R''_i \), and then from \( R''_i \) to \( R_i \); since the rule is weakly private-side uncompromising for both components of agent \( i \)'s bundle, \( \Omega = \Omega'' \) and \( x_i = x''_i \). But then, since the rule is public-side neutral for the private component of agent \( i \)'s bundle, \( x'_i = x''_i = x_i \). But then, for the agent with preferences \( R''_i \) not to benefit from announcing \( R_i \) instead, it must be that \( \Omega = \Omega'' = \Omega' \).

- if \( x_i = p(R''_i) = p(R''_i) \) and \( \Omega = p(R''_i) = p(R''_i) \), then \( \Omega' = \Omega' \) and \( x'_i = x_i \) for strategy-proofness, since otherwise the agent with preferences \( R''_i \) would benefit from announcing \( R_i \) instead.

Therefore, the rule is weakly peak-only and a strategy-proof rule is weakly peak-only if and only if it is weakly uncompromising. ■

**Lemma 3** A rule is strategy-proof and weakly peak-only if and only if it has a closed range.

**Proof.** Step 1: We first prove that a strategy-proof rule that is weakly peak-only has a closed range. Assume not. Then, for some \( R \in \mathbb{R}^N \), some \( i \in N \), one of the following does not hold:

- if \( p(R''_i) = 0 \), the rule chooses \( x_j \) for \( R_i \); if \( p(R''_i) = E \), the rule chooses \( x_i \geq x_j \) for \( R_i \); and for any \( p(R''_i) \), the rule chooses \( x_i = \text{med} \{ p(R''_i), x_j, \bar{x}_i \} \) for \( R_i \). Assume this does not hold. First, we can establish that all announcements of a preference relation such that \( p(R''_i) = 0 \) must lead to a unique private amount \( x_i \); if any two such announcements led to two different amounts of the private good, the agent with the greatest private amount would announce the other agent’s preferences instead and strategy-proofness would be violated. Similarly, strategy-proofness suffices to show that all announcements of a preference relation such that \( p(R''_i) = E \) must lead to a unique private amount \( \bar{x}_i \). We can now establish that \( x_i \leq \bar{x}_i \). Suppose, by contradiction, that \( x_i > \bar{x}_i \). Then an agent with preferences such that \( p(R''_i) = 0 \) would rather announce \( R_i \), where \( p(R''_i) = E \) and strategy-proofness would be violated. There are then two possible cases:

\[-p(R''_i) \in [x_i, \bar{x}_i] \text{ and therefore } \text{med} \{ p(R''_i), x_i, \bar{x}_i \} = p(R''_i) \]; without loss of generality, assume that \( x_i < p(R''_i) \). Then we can find a preference relation \( R''_i \)
with \( p(R_i^e) = p(R_i^f) \) and \( p(R_i^{0\Omega}) = p(R_i^{0\Omega}) \) such that \( x_i < r_i^e(\overline{x}_i) \leq p(R_i^f) = p(R_i^f) \). Since the rule is weakly peak-only, if agent \( i \)'s preference relation were \( R_i^f \) instead of \( R_i \), the rule would still pick \( x_i' = x_i \) and \( \Omega' = \Omega \). But then \( x_i = x_i' < r_i^f(\overline{x}_i) \leq p(R_i^f) = p(R_i^f) \) and an agent with preferences \( R_i^f \) would benefit from announcing a preference relation \( \overline{R}_i \) such that \( p(R_i^f) = E \) instead and the rule would not be strategy-proof.

\[-p(R_i^f) \notin [\overline{x}_i, \overline{x}_i] ; \text{ without loss of generality, assume that } p(R_i^f) < x_i \text{ and therefore, } \text{med} \{p(R_i^f), x_i, \overline{x}_i\} = x_i \}; \text{ again, without loss of generality, assume } x_i < x_i. \text{ Then } 0 \leq x_i < x_i \text{ and an agent with preferences } R_i \text{ such that } p(R_i^f) = 0 \text{ would rather announce } R_i, \text{ and the rule would not be strategy-proof.} \]

Therefore, \( x_i = \text{med} \{p(R_i^f), x_i, \overline{x}_i\} \).

\[• \text{for all } x_i \in [\overline{x}_i, \overline{x}_i], \text{ if } p(R_i^f) \text{ is such that the rule chooses } x_i \text{ for } (R_i, R_{-i}), \text{ if } p(R_i^f) = 0, \text{ the rule chooses } \Omega_1(x_i) \text{ for } R; \text{ if } p(R_i^{0\Omega}) = E, \text{ the rule chooses } \Omega_1(x_i) \text{ for } R; \text{ and for any } p(R_i^{0\Omega}), \text{ the rule chooses } \Omega = \text{med} \{p(R_i^f), \Omega_1(x_i), \overline{\Omega}_1(x_i)\} \text{ for } R. \text{ Assume this does not hold. First, we can establish that, for any such } x_i, \text{ all announcements of a preference relation where } p(R_i^f) \text{ is such that } x_i \text{ is chosen by the rule, and } p(R_i^{0\Omega}) = 0, \text{ lead to a unique public good level of } \Omega_1(x_i); \text{ if any two such announcements led to two different amounts of the public good, the agent that was faced with the greatest public good level would benefit from announcing the other agent’s preferences instead and strategy-proofness would be violated. Similarly, strategy-proofness suffices to show that for any such } x_i, \text{ all announcements of a preference relation where } p(R_i^f) \text{ is such that } x_i \text{ is chosen by the rule, and } p(R_i^{0\Omega}) = E \text{ lead to a unique public good level of } \overline{\Omega}_1(x_i). \text{ We can now establish that } \Omega_1(x_i) \leq \overline{\Omega}_1(x_i). \text{ Suppose, by contradiction, that } \Omega_1(x_i) > \overline{\Omega}_1(x_i). \text{ Then an agent with preferences where } p(R_i^f) \text{ is such that } x_i \text{ is chosen by the rule, and } p(R_i^{0\Omega}) = 0, \text{ would benefit from announcing } \overline{R}_i = (R_i^f, R_i^{0\Omega}) \text{ where } p(R_i^{0\Omega}) \text{ is still such that } x_i \text{ is chosen by the rule and } p(R_i^{0\Omega}) = E, \text{ and strategy-proofness would be violated. There are then two possible cases: }

\[-p(R_i^{0\Omega}) \in [\Omega_1(x_i), \overline{\Omega}_1(x_i)] \text{ and therefore } \text{med} \{p(R_i^{0\Omega}), \Omega_1(x_i), \overline{\Omega}_1(x_i)\} = p(R_i^{0\Omega}); \text{ without loss of generality, assume that } \Omega < p(R_i^{0\Omega}). \text{ Then we can find a preference relation } R_i' \text{ such that } p(R_i'^f) = p(R_i^f), p(R_i'^{0\Omega}) = p(R_i^{0\Omega}) \text{ and } \Omega < r_i^{0\Omega}(\overline{\Omega}_1(x_i)) \leq p(R_i'^{0\Omega}) = p(R_i^{0\Omega}). \text{ Since the rule is weakly peak-only, if agent } i \text{'s preference relation were } R_i' \text{ instead of } R_i, \text{ the rule would still pick } x_i' = x_i \text{ and } \Omega' = \Omega. \text{ But then } \Omega = \Omega' < r_i^{0\Omega}(\overline{\Omega}_1(x_i)) \leq p(R_i'^{0\Omega}) = p(R_i^{0\Omega}) \text{ and an agent with preferences } R_i' \text{ would benefit from announcing } \overline{R}_i = (R_i^f, R_i^{0\Omega}) \text{ where } p(R_i^{0\Omega}) \text{ is still such that } x_i \text{ is chosen by the rule and } p(R_i^{0\Omega}) = E, \text{ and the rule would not be strategy-proof.} \]
\[ p(R^{0}_{i}) \notin \left[ \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \right] \]; without loss of generality, assume that \( p(R^{0}_{i}) < \underline{\Omega}(x_{i}) \) and therefore, \( med \{ p(R^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} = \underline{\Omega}(x_{i}) \); again, without loss of generality, assume \( \Omega < \underline{\Omega}(x_{i}) \). Then \( 0 \leq \Omega < \underline{\Omega}(x_{i}) \) and an agent with preferences \( R = (R^{w}_{i}, R^{0}_{i}) \) where \( p(R^{w}_{i}) \) is still such that \( x_{i} \) is chosen by the rule and \( p(R^{0}_{i}) = 0 \) would benefit from announcing \( R_{i} \) instead, and the rule would not be strategy-proof.

Therefore, \( \Omega = med \{ p(R^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} \).

Therefore, a strategy-proof rule that is weakly peak-only has a closed range.

Step 2: We now want to prove the reverse implication: that a rule that has a closed range is weakly peak-only and strategy-proof. We first establish that it must be weakly peak-only. We need to check that, for all \( i \in N \), and all \( R, R' \in \mathbb{R}^{N}_{sp, lex} \) such that \( R_{-i} = R'_{-i} \), \( p(R^{w}_{i}) = p(R^{w'}_{i}) \) and \( p(R^{0}_{i}) = p(R'^{0}_{i}) \), the rule chooses \( \Omega \) and \( x_{i} \) for \( R \) and \( \Omega' \) and \( x_{i}' \) for \( R' \) where \( \Omega = \Omega' \) and \( x_{i} = x_{i}' \). Since the rule has a closed range, \( x_{i} = med \{ p(R^{w}_{i}), x_{i}, \overline{\Omega}(x_{i}) \} \) and \( x_{i}' = med \{ p(R'^{w}_{i}), x_{i}', \overline{\Omega}(x_{i}) \} \), and with \( p(R^{w}_{i}) = p(R'^{w}_{i}) \), we have \( x_{i} = x_{i}' \). Also, since \( x_{i} \) and \( x_{i}' \) are the same for \( R_{i} \) and \( R'_{i} \), then \( \Omega = med \{ p(R^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} \) and \( \Omega' = med \{ p(R'^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} \), and with \( p(R^{0}_{i}) = p(R'^{0}_{i}) \), we have \( \Omega = \Omega' \). Therefore, the rule is weakly-peak-only.

We also want to prove that it is strategy-proof. Assume not, towards a contradiction. Then some agent would benefit from misrepresenting his preferences in the associated direct revelation game. Let \( R_{i} \) denote his true preference relation and \( R'_{i} \) represent an alternative announcement by agent \( i \); \( (\Omega, x) \) is the choice made by the rule for \( R = (R_{i}, R_{-i}) \) and \( (\Omega', x') \) is the choice made by the rule for \( R' = (R'_{i}, R_{-i}) \).

If agent \( i \) gets his private peak at \( \Omega \), since the rule has a closed range, we know that \( x_{i} = med \{ p(R^{w}_{i}), x_{i}, \overline{\Omega}(x_{i}) \} \) and therefore \( p(R^{0}_{i}) \in [x_{i}, \overline{\Omega}(x_{i})] \); if instead agent \( i \) does not get his private peak at \( \Omega \), assume without loss of generality, that \( p(R^{w}_{i}) < x_{i} \); since the rule has a closed range, we know that \( x_{i} = med \{ p(R^{w}_{i}), x_{i}, \overline{\Omega}(x_{i}) \} = x_{i} \) and it is not possible that \( x_{i}' < x_{i} \) regardless of the announcement \( R'_{i} \) since \( med \{ p(R'^{w}_{i}), x_{i}, \overline{\Omega}(x_{i}) \} \geq x_{i} \) for all \( p(R'^{w}_{i}) \). In either case, if \( \Omega = p(R^{0}_{i}) \), he cannot be made better off. If \( \Omega \neq p(R^{0}_{i}) \), assume without loss of generality that \( p(R^{0}_{i}) < \Omega \); then, he can only be made better off at an allocation that gives him \( x_{i}' = x_{i} \) and \( \Omega' < \Omega \). Since the rule has a closed range, by announcing \( R'_{i} \) such that \( x_{i}' = x_{i} \), we have \( \Omega = med \{ p(R^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} = \underline{\Omega}(x_{i}) \) since \( p(R^{0}_{i}) < \Omega \), but we also have that \( \Omega' = med \{ p(R'^{0}_{i}), \underline{\Omega}(x_{i}), \overline{\Omega}(x_{i}) \} \geq \underline{\Omega}(x_{i}) \) for all \( p(R'^{0}_{i}) \). Therefore, regardless of the agent’s announcement, he cannot be made better off.

Therefore, no agent can benefit from misrepresenting his preferences and a rule with a closed range is strategy-proof.
Proof of Proposition 1  Proposition 1 follows from Lemmas 1-3.

We now reintroduce efficiency and another mild anonymity requirement that is meant to rule out dictatorship\textsuperscript{13}: that the rule give identical treatment to agents that have identical preferences:

**Definition 9**  A rule is symmetric if for all $R \in \mathbb{R}_{sp,lex}^N$ and for all $i, j \in N$, if $R_i = R_j$, the rule chooses $(\Omega, x)$ for $R$ where $x_i I_i x_j$.

In order to proceed with the analysis of strategy-proofness in this setting, let us first recall the notion of a uniform distribution.

**Definition 10**  If whenever $\Omega \leq \sum_{i \in N} p(R_i^x)$, $x_i = \min \{p(R_i^x), \lambda\}$ for all $i \in N$, and whenever $\Omega \geq \sum_{i \in N} p(R_i^x)$, $x_i = \max \{p(R_i^x), \lambda\}$ for all $i \in N$, where $\lambda$ is determined by feasibility, then $x$ is a uniform distribution of $\Omega$, denoted by $x = U(\Omega)$.

Our next result\textsuperscript{14} reinforces the prominence of a uniform distribution in a rule for the mixed-good model:

**Theorem 1**  In a mixed-good economy, any rule that satisfies efficiency, strategy-proofness, continuity and symmetry must use a uniform distribution.

**Proof.**  If for some $R \in \mathbb{R}_{sp,lex}^N$, the rule picks the sum of the private peaks as the public good level, efficiency alone is enough to ensure that each agent gets his private peak - and this coincides with a uniform distribution. Assume now, without loss of generality that for some $R \in \mathbb{R}_{sp,lex}^N$, the rule picks $\Omega < \sum_{i \in N} p(R_i^x)$. For efficiency, $x_i \leq p(R_i^x)$ for all $i \in N$ and for feasibility $x_j < p(R_j^x)$ for some $j \in N$. For symmetry, any two agents with the same preferences get the same private amount. Towards a contradiction, assume $\Omega$ is not assigned to agents according to a uniform distribution. In order to arrive at a contradiction, we proceed with what we name the 'Bracing algorithm' (Fig.1):

**Step 1**  We start Step 1 with the initial $\Omega$ and some distribution that we denote by $x$ and that is not a uniform distribution of $\Omega$, but $x_i \leq p(R_i^x)$ for all $i \in N$.

\textsuperscript{13}Given that each agent cares only about the public good level and the private amount he receives, dictatorship here encompasses any rule that always gives one agent his peaks, if feasible, regardless of how the private amounts that the other agents receive are determined.

\textsuperscript{14}Ching (1994) proves a similar result for the unidimensional private-good model.
Figure 1: This is a simple example that illustrates the logic of the bracing algorithm: assume agent 3’s public peak is at $\Omega$ and, for simplicity of notation, let $p_i$ represent agent $i$’s private peak. Then, if we move agent 2’s preferences in the direction of agent 3’s preferences, and since the rule is weakly uncompromising, agent 2’s bundle must remain the same; however, symmetry implies that at the point where agents 2 and 3 have the exact same preferences, they must receive the same amount of the private good. Therefore, at that point, agent 3 and agent 2 both receive $x_2$ and efficiency can only hold if $x_1 \leq p_1$. But then, since $\Omega$ must remain the same throughout, feasibility is violated.

Take the agent $l$ such that $x_l = \max_i \{x_i\}$; if several agents receive $x_l$, let $l$ be the one with the highest private peak; if there are several agents that receive $x_l$ with that same highest private peak, let $l$ be an arbitrary choice from that set. If agent $l$’s public peak is $\Omega$, go to Step 2; otherwise, let agent $l$’s preferences move to $R_0^l$ such that $R_{x_0}^l = R_{x_l}^l$ and $p(R_{x_0}^l) = \Omega$. From Proposition 1, we know that the rule is weakly uncompromising and it must then choose the same $x_0^l = x_l$ and $\Omega' = \Omega$ for $R' = (R_0^l, R_{-l})$. Naturally, for symmetry, if any other agent has the same preferences as $R_0^l$, he must also get a private good amount $x_0^l = x_l$. Also, the distribution of $\Omega' = \Omega$ is still such that $x_0^l \leq p(R_{x_0}^l)$ for all $i \in N$, since the rule is efficient and $\Omega' < \sum_{i \in N} p(R_{x_0}^l)$ but it is still not the uniform: since $x \neq U(\Omega)$, we had $x_i \leq \min \{p(R_{x_0}^l), x_l\}$ for all $i \in N$ with strict inequality for at least one agent $j$; then, if $x' = U(\Omega')$ and $x_0^l = x_l$, we would have $x_i' = p(R_{x_0}^l)$ for all $i$ such that $p(R_{x_0}^l) \leq x_i$ and $x_i' \geq x_l$ for all $i$ such that $p(R_{x_0}^l) > x_i$; but then $x_i' \geq x_i$ for all $i$ and $x_j' > x_j$ for at least one agent $j$, contradicting $\Omega' = \Omega$.

Now either $x_0^l = x_l = \max_i \{x_i'\}$ and if several agents receive $x_0^l$, $l$ is still the one with the highest private peak and we move to Step 2, or we repeat Step 1. The repetition of Step 1 is, however, finite, since there is a finite number of agents - and, therefore, a finite number of agents with public peaks different from $\Omega$.

$\Omega$ is the same throughout the possible repetitions of Step 1 and the final distribution is never the uniform. Also, at the end, the agent with the highest private amount has a public peak of $\Omega$. For the next step, let that
agent be agent \( l \).

**Step 2** We start Step 2 with the initial \( \Omega \) and some distribution that we denote by \( x \) where \( x \neq U(\Omega) \) and \( x_i \leq p(R^{(i)}_x) \) for all \( i \in N \).

Letting \( S = \{ i \in N \mid x_i < p(R^{(i)}_x) \} \), choose agent \( k \) such that \( x_k = \min_{i \in S} \{ x_i \} \); if several agents receive \( x_k \), let \( k \) be the one with the highest private peak.

Again, if there are several agents that receive \( x_k \), let \( k \) be an arbitrary choice from that set. We know that \( x_k < x_l \), given that \( x \) is not a uniform distribution of \( \Omega \). Now let agent \( k \)'s preferences move from \( R_k \) to \( R'_k = R_l \). Since the rule is weakly uncomprising and weakly peak-only, \( x_k < p(R^{(k)}_k) \), \( x_k < x_l \leq p(R^{(l)}_l) = p(R^{(k)}_k) \) and \( p(R^{(l)}_k) = p(R^{(1)}_k) = \Omega \), then the rule must choose the same \( x'_k = x_k \) and \( \Omega' = \Omega \) for \( R' = (R'_k, R_{-k}) \). For symmetry to hold, we must have \( x'_l = x'_k = x_k < x_l \) - and if any other agent has the same preferences as \( R_k \), he must also get the same private good amount i.e. letting \( L' = \{ i \in N \mid R'_i = R'_l = R_l \} \), we have \( x'_i = x'_k = x'_l < p(R^{(l)}_i) \) for all \( i \in L' \).

Since \( \Omega' = \Omega \), \( x'_i = x'_k = x'_l < p(R^{(l)}_i) = p(R^{(k)}_i) \) and the rule is efficient, then \( x'_i \leq p(R^{(k)}_i) \) for all \( i \in N \) and one of the following occurs:

- **a)** \( \sum_{i \in N \setminus L'} p(R^{(i)}_i) + |L'| \cdot x'_l < \Omega' \) i.e. giving all the agents in \( L' \) (including \( l \) and \( k \)) the amount \( x'_l \) and all the other agents their private peaks does not exhaust \( \Omega' \) and feasibility is violated and we reach a contradiction (Fig.2).

- **b)** \( \sum_{i \in N \setminus L'} p(R^{(i)}_i) + |L'| \cdot x'_l \geq \Omega' \) but the distribution of \( \Omega' \) is still not the uniform; since \( x_k = \min_{i \in S} \{ x_i \} \), agent \( k \) is getting the same amount at \( \Omega \) and \( \Omega' \), and \( l \) (and any other element in \( L' \)) was getting \( x_l \) at \( \Omega \) and is getting \( x'_l < x_l \) at \( \Omega' \), then at least one agent \( j \) who was not getting his private peak at \( \Omega \) must now get \( x'_j > x_j \) - all the agents who were getting their private peaks at \( \Omega \) cannot get more at \( \Omega' \) for efficiency. But since \( j \in S \), \( x'_j > x_j \geq x_k = x'_k \), and \( x'_k < p(R^{(k)}_k) \), the distribution of \( \Omega' \) is not the uniform. Then, one of the following occurs:

- **b1)** there is an agent \( i \) such that \( p(R^{(l)}_i) \geq x'_l > p(R^{(k)}_i) = p(R^{(l)}_i) \) and we go back to Step 1 in order to restart the accumulation at a higher private peak; again, there is a finite number of agents and this step can only be reached a finite number of times, at most until the highest private peak is reached as the accumulation point.

- **b2)** there is no agent \( i \) such that \( p(R^{(l)}_i) \geq x'_l > p(R^{(k)}_i) = p(R^{(l)}_i) \), \( S' \setminus L' \) is non-empty and there is an agent \( i \in S' \setminus L' \) such that \( x'_l \leq x'_i = x'_k \) (Fig.3).

We then move to Step 3.1.

- **b3)** there is no agent \( i \) such that \( p(R^{(l)}_i) \geq x'_l > p(R^{(k)}_i) = p(R^{(l)}_i) \), \( S' \setminus L' \) is non-empty and for all \( i \in S' \setminus L' \) we have that \( x'_l > x'_i = x'_k \) and, letting
Figure 2: This is an illustration of Case a) in Step 2. We begin with a distribution of the initial $\Omega$ that is not uniform. Out of the agents not getting their private peaks, agent 3 is the one who receives the smallest amount. We let agent 3’s preferences move from $R_3$ to $R'_3 = R_5$. Since the rule is weakly uncompromising and weakly peak-only, the rule must still choose the same $x'_3 = x_3$ and $\Omega' = \Omega$ for $R' = (R'_3, R_{-3})$. For symmetry to hold, we must also have $x'_5 = x'_3 = x_3 < x_5$. For efficiency to hold, and given that agents 3 and 5 are getting less than their private peaks, no agent can get more than his private peak. But then since $\Omega$ must remain the same, and $x_4$ can at most increase to $p_4$, the sum of the private amounts cannot equal $\Omega$ and feasibility is violated.

Figure 3: This is an illustration of Case b2) in Step 2: we perform the same manipulation as in Figure 2 but do not yet reach a contradiction; also, no agent starts receiving a private amount greater than $p_5$, $L' = \{3, 5\}$ and $S' \setminus L' = \{2, 4\}$ is non-empty; moreover, agent 2 is in $S' \setminus L'$ and $x'_2 \leq x'_3 = x'_5$. We then move to Step 3.1.
$x_M = \max_{i \in N} \{x_i\}$, at least one of the agents getting $x_M$ is not getting his private peak (Fig.4). We then move to Step 3.2.

-b4) there is no agent $i$ such that $p(R_i^{x'}) \geq x_i' > p(R_i^{x''}) = p(R_i^{x''})$, and either $S' \setminus L'$ is empty, or $S' \setminus L'$ is non-empty and for all $i \in S' \setminus L'$ we have that $x_i' > x_i' = x_k'$ and, letting $x_M = \max_{i \in N} \{x_i\}$, all the agents getting $x_M$ are getting their private peaks (Fig.5). We then move to Step 3.3.

**Step 3.1** We start Step 3.1 (Fig.6) with the initial $\Omega$ and a distribution that we now denote by $x$ where $x \neq U(\Omega)$ and $x_i \leq p(R_i^x)$ for all $i \in N$. Also, we have an accumulation set $L = \{i \in N \mid R_i = R_i\}$ and $x_j = x_l$ for all $j \in L$. There is no agent $i$ such that $x_i > p(R_i^x)$, and letting $S = \{i \in N \mid x_i < p(R_i^x)\}$, there is an agent $i \in S' \setminus L$ such that either $x_i \leq x_l < p(R_l^x)$ or $x_i < x_l \leq p(R_l^x)$.

Take the agent $h$ such that $x_h = \min_{i \in S \setminus L} \{x_i\}$; if several agents in $S \setminus L$ receive $x_h$, let $h$ be the one with the highest private peak. Again, if there are several agents that receive $x_h$ with that same highest private peak, let $h$ be an arbitrary choice from that set. We know that $x_h \leq x_l = x_j$ for all $j \in L$. Now let agent $h$’s preferences move from $R_h$ to $R_h = R_l(= R_l$ for all $i \in L$). Since the rule is weakly compromising and weakly-peak-only, $x_h < p(R_h^x)$, $x_h < p(R_l^x) = p(R_l^{\Omega})$ and $p(R_l^{\Omega}) = p(R_l^{\Omega}) = \Omega$, then the rule must choose the same $x_i' = x_h$ and $\Omega' = \Omega$ for $R'' = (R_h', R_l)$. For symmetry to hold, we must have $x_i' = x_h'$ and $x_i' \leq x_i - 1$ and all the agents in $L' = L \cup \{h\}$ get a private good amount $x_h' = x_h$.

Since $\Omega' = \Omega$, $x_j' = x_h' = x_h < p(R_i^{x''})$ for all $j \in L'$, and the rule is efficient, then $x_i' \leq p(R_i^{x''})$ for all $i \in N$ and either case a) occurs and we stop or case b) occurs. If b) occurs and we initially had $x_h < x_l(= x_j$ for all $j \in L$) we know, by the argument stated in Step 2, that $x' \neq U(\Omega')$ and we move to the corresponding step of the relevant subcase of b). If b) occurs and we initially had $x_h = \min_{i \in S \setminus L} \{x_i\} = x_l < p(R_i^x)$, then we had $x_i = p(R_i^x)$ for all $i$ such that $p(R_i^x) \leq x_l$, and $x_i \geq x_l$ for all $i$ such that $p(R_i^x) > x_l$, with $x_i > x_l$ for at least one such $i$ since $x \neq U(\Omega)$; then, we must have at least one agent $i$ such that $x_i' > x_i' = x_l$ at $\Omega'$ since otherwise we would have $x_i' \geq x_i$ for all $i$ with strict inequality for at least one $i$, contradicting $\Omega' = \Omega$. Therefore, if initially $x_h = x_l(= x_l$ for all $i \in L$), we also know that $x' \neq U(\Omega')$ and we can then move to the corresponding step of the relevant subcase of b).

The purpose of this step is to continue the accumulation process from the previous step(s). Out of the agents not getting their private peaks, we choose the one that receives the lowest private amount and move his preferences to the ”accumulation preferences” $R_l$ and therefore force the ”accumulation point” $x_l$ to decrease to that lowest private amount (or at most be kept.
Figure 4: This is an illustration of Case b3) in Step 2: we perform the same manipulation as in Figure 2 but do not yet reach a contradiction; also, no agent starts receiving a private amount greater than $p_5$, $L' = \{3, 5\}$ and $S \setminus L' = \{4\}$ is non-empty; however, agent 4 receives $x'_4 > x'_3 = x'_5$. Since agent 4 receives the maximum private amount but not his private peak, we move to Step 3.2.

Figure 5: This is an illustration of Case b4) in Step 2: we perform the same manipulation as in Figure 2 but do not yet reach a contradiction; also, no agent starts receiving a private amount greater than $p_5$, $L' = \{3, 5\}$ and $S \setminus L'$ is empty - agent 4 who receives the maximum private amount is at his private peak. We then move to Step 3.3.
constant at that lowest amount) so that we now add one more agent to the "accumulation set" \((L' = L \cup \{ h \})\) and therefore \(|L'| = |L| + 1\) at a possibly even lower private amount, and so that we can force the others to increase as much as possible with the ultimate goal of achieving an incompatibility between efficiency and feasibility.

**Step 3.2** We start Step 3.2 with the initial \(\Omega\) and a distribution that we now denote by \(x\) where \(x \neq U(\Omega)\) and \(x_i \leq p(R_i^\circ)\) for all \(i \in N\). Also, we have an accumulation set \(L = \{ i \in N \mid R_i = R_l \}\) and \(x_j = x_l < p(R_i^\circ) = p(R_i^\circ)\) for all \(j \in L\). There is no agent \(i\) such that \(x_i > p(R_i^\circ)\), and letting \(S = \{ i \in N \mid x_i < p(R_i^\circ) \}\), \(S \setminus L\) is non-empty and for all \(i \in S \setminus L\), we have that \(x_i > x_l = x_j\) for all \(j \in L\). Also, letting \(x_M = \max_{i \in N} \{ x_i \} \), at least one of the agents getting \(x_M\) is not getting his private peak.

Out of the agents that receive \(x_M\), let \(g\) be the one with the highest private peak. Again, if there are several agents that receive \(x_M\) with that same highest private peak, let \(g\) be an arbitrary choice from that set. We know that \(p(R_g^x) > x_M = x_g\). Then \(g \in S \setminus L\) and \(x_g > x_l = x_j\) for all \(j \in L\). Also since for every agent \(i\), \(x_i \leq p(R_i^\circ)\), then for all \(j \in L\), \(x_j = x_l < x_g \leq p(R_i^\circ) = p(R_i^\circ)\). Now let agent \(g\)'s preferences move from \(R_g\) to \(R_g' = R_j\) for all \(j \in L\). Since the rule is weakly uncompromising and weakly peak-only, \(x_g < p(R_g^x)\), \(x_g \leq p(R_i^\circ)\) and \(p(R_g^\circ) = p(R_i^\circ) = \Omega\), then the rule must
Figur 7: This is an illustration of Step 3.2, following the case in Figure 4. Initially, \( L = \{3, 5\} \) and \( S \setminus L = \{4\} \) is non-empty but \( x_4 > x_3 = x_5 \). We let agent 4’s preferences move from \( R_4 \) to \( R'_4 = R_5 = R_3 \). Since the rule is weakly uncompromising and weakly peak-only, the rule must still choose the same \( x'_4 = x_4 \) and \( \Omega' = \Omega \) for \( R' = (R'_4, R_{-4}) \). For symmetry to hold, we must also have \( x'_4 = x'_3 = x'_5 > x_5 = x_3 \). But then since \( \Omega \) must remain the same, the private amounts of agents 3 and 5 increase, agent 4’s does not change, and \( x_1 \) and \( x_2 \) can at most decrease to 0, the sum of the private amounts cannot equal \( \Omega \) and feasibility is violated.

choose the same \( x'_g = x_g \) and \( \Omega' = \Omega \) for \( R'_g = (R'_g, R_{-g}) \). For symmetry to hold, we must have \( x'_l = x'_g = x_g > x_l \) and all the agents in \( L' = L \cup \{g\} \) get a private good amount \( x'_g = x_g \).

Since \( \Omega' = \Omega \) and \( x'_j = x'_g = x_g > x_j \) for all \( j \in L' \), then one of the following must occur:

- \( |L'| \cdot x'_l > \Omega' \) and giving all the agents in \( L' \) (including \( l, k \) and \( g \)) the amount \( x'_l \) and all the other agents a non-negative amount is not feasible for \( \Omega' \) and we reach a contradiction (Fig.7).

- \( |L'| \cdot x'_l \leq \Omega' \) and at least one agent \( i \notin L' \) must get less of the private good at \( \Omega' \); since \( x_i \leq x_g, x_i \leq p(R^*_i) \) and \( x'_i < x_i \), then \( x'_i < p(R^*_i) = p(R^*_j) \) and \( x'_i < x_g = x_j \) for \( j \in L' \). Clearly, this is not a uniform distribution of \( \Omega' \). Now, if there is an agent \( i \) such that \( p(R^*_i) \geq x'_i > p(R^*_j) \) for all \( j \in L' \), we go back to Step 1 in order to restart the accumulation at a higher private peak; again, there is a finite number of agents and this step can only be reached a finite number of times, at most until the highest private peak is reached as the accumulation point. Otherwise (Fig.8), we can proceed to Step 3.1. again (Fig.9).

The purpose of this step is to continue the accumulation process from the previous steps. Out of the agents not getting their private peaks, we choose
Figure 8: This is another illustration of Step 3.2, following the case in Figure 4, where we perform the same manipulation as in Figure 7 but do not yet reach a contradiction; also, no agent starts receiving a private amount greater than $p_5$, $L' = \{3, 4, 5\}$ and $S' \setminus L' = \{2\}$ is non-empty; moreover, agent 2 is in $S' \setminus L'$ and $x'_2 < x'_3 = x'_5$. We then move to Step 3.1. with one more agent in the accumulation set.

Figure 9: Following the case in Figure 8, we proceed with the algorithm by reapplying Step 3.1. By moving agent 2 to the accumulation set, we know that his private amount must remain the same and all the other agents in the accumulation set must receive $x_2$ as well, for symmetry. But then efficiency and feasibility are again incompatible.
the one that receives the highest private amount and move his preferences to the "accumulation preferences" $R_i$ and therefore force the "accumulation point" $x_l$ to increase to that highest private amount so that we now add one more agent to the "accumulation set" ($L' = L \cup \{g\}$ and therefore $|L'| = |L| + 1$) at a higher private amount, and so that we can force the others to decrease as much as possible with the ultimate goal of achieving an incompatibility between efficiency and feasibility.

**Step 3.3** We start Step 3.3 with the initial $\Omega$ and a distribution that we now denote by $x$ where $x \neq U(\Omega)$ and $x_i \leq p(R_i^f)$ for all $i \in N$. Also, we have an accumulation set $L = \{i \in N \mid R_i = R_l\}$ and $x_j = x_l < p(R_j^f) = p(R_l^f)$ for all $j \in L$. There is no agent $i$ such that $x_i > p(R_i^f)$, and letting $S = \{i \in N \mid x_i < p(R_i^f)\}$, either $S \setminus L$ is empty and, letting $x_M = \max_{i \in N} \{x_i\}$, $x \not\in U(\Omega)$ implies that all the agents getting $x_M > x_l$ are getting their private peaks; or $S \setminus L$ is non-empty and for all $i \in S \setminus L$ we have that $x_i > x_l = x_j$ for all $j \in L$ and, letting $x_M = \max_{i \in N} \{x_i\}$, all the agents getting $x_M$ are getting their private peaks.

In either case, for all $i$ such that $x_i = x_M$, $p(R_i^f) = x_M$. If neither of these agents have a public peak of $\Omega$, we return to Step 1 and this case can only be reached a finite number of times since there is a finite number of agents and throughout the algorithm no agent with a public peak of $\Omega$ changes his public peak to any other amount. If there is a non-empty set of agents that get $x_M$ at $\Omega$ and for whom $x_M$ is the private peak and $\Omega$ is the public peak, then let $f$ be an arbitrary choice from that set. We know that $x_f = x_M = p(R_i^f) \leq p(R_j^f) = p(R_l^f)$ for all $j \in L$, and $\Omega = p(R_i^\Omega) = p(R_j^\Omega) = p(R_l^\Omega)$ for all $j \in L$.

Now take a sequence $R_i^v \to R_j$ where $v \in [0, 1] \cap Q$ and $r_i^{vf} = (1 - v).r_i^f + v.r_i^\Omega$ and $r_i^{ol} = (1 - v).r_i^\Omega + v.r_i^f$ which means that if $p(R_j^f) < p(R_i^f)$, then $p(R_i^{vf})$ is decreasing in $v$\textsuperscript{15}, if $p(R_j^\Omega) = p(R_i^f)$, then $p(R_i^{vf}) = p(R_i^\Omega)$ throughout the sequence, $p(R_i^{ol}) = \Omega$ throughout the sequence, and $R_i^0 = R_i$ and $R_i^1 = R_j$. Since the rule is weakly peak-only and weakly uncompromising, and $x_i < x_f = p(R_j^f) \leq p(R_i^{vf}) \leq p(R_i^\Omega)$, then throughout the sequence $x_i^v = x_l$ and $\Omega^v = \Omega$. At the limit, symmetry implies that for $R_i^1 = R_i^1 = R_j$, the rule chooses $x_f^1 = x_i^1 = x_l$ and $\Omega^1 = \Omega$. Then, for continuity, we can find a $v$ and the corresponding $R_i^v$ such that for

\textsuperscript{15}Looking at $r_i^{vf} = (1 - v).r_i^f + v.r_i^\Omega$ where $v \in [0, 1]$, $x = (1 - v).r_i^f(x) + v.r_i^\Omega(x)$ implicitly defines $p(R_i^{vf})$. Then,

$$\frac{dx}{dv}(x) = \frac{r_i^f(x) - r_i^\Omega(x)}{1 - v} \cdot \frac{dr_i^f(x)}{dx} - (1 - v) \cdot \frac{dr_i^\Omega(x)}{dx} < 0$$

for all $x \in \left[p(R_j^f), p(R_i^f)\right]$. Therefore, $p(R_i^{vf})$ is strictly decreasing in $v$ for $v \in [0, 1] \cap Q$. 27
\( R^v = (R_l^v, R_{-l}) \), the rule chooses \( \Omega = \Omega', x_i^v = x_l \) and \( x_f^v = x_f - \varepsilon \) where \( \varepsilon > 0 \) is arbitrarily small and such that \( \varepsilon < \frac{|I|}{|N|} (x_f - x_l) \) (Fig.10).

After moving agent \( l \)'s preferences from \( R_l \) to \( R_l^v \), let agent \( f \)'s preferences move from \( R_f \) to \( R_f' = R_f^v \). Since the rule is weakly uncompromising and weakly peak-only, \( x_f^v < p(R_f^v) \leq p(R_f^{\Omega}) \) and \( p(R_f^{\Omega}) = p(R_f^{\Omega''}) = \Omega \), it must choose for \( R'' = (R_l^v, R_f' = R_f^v, R_{-l-f}) \) a \( \Omega' = \Omega \) and \( x_f' = x_f^v \). For symmetry, \( x_i' = x_i^v = x_f^v = x_f - \varepsilon \). Then, \( L' = \{i \in N \mid R_i = R_i^v\} \) and either \(|L'| \cdot x_i' > \Omega' \) and giving all the agents in \( L' \) (including \( l \) and \( k \)) the amount \( x_i' \) and all the other agents a non-negative amount is not feasible for \( \Omega' \) and we reach a contradiction (Fig.11); or \(|L'| \cdot x_i' \leq \Omega' \) and we go on with the step (Fig.12).

If a contradiction has not yet been reached, now let agent \( l \)'s preferences move from \( R_l^v \) to the original \( R_l (= R_l^u) \). Since the rule is weakly uncompromising and weakly peak-only, \( x_i' < p(R_i^v) \leq p(R_i^{\Omega}) = p(R_i^{\Omega''}) = \Omega \), it must choose for \( R'' = (R_l' = R_l^v, R_{-l-f}) \) a \( \Omega'' = \Omega \) and \( x_f'' = x_i^v = x_f - \varepsilon \) and all the agents in \( L'' = L \) get a private good amount equal to \( x_f - \varepsilon \).

Since \( \Omega'' = \Omega \) and for \( \varepsilon \) small enough, \( x_j = x_f - \varepsilon > x_l \) for all \( j \in L'' \), then one of the following must occur:

- \(|L''| \cdot x_i'' > \Omega'' \) and giving all the agents in \( L'' \) (including \( l \) and \( k \)) the amount \( x_i'' \) and all the other agents a non-negative amount is not feasible for \( \Omega'' \) and we reach a contradiction.

- \(|L''| \cdot x_i'' \leq \Omega'' \) and at least one agent \( i \notin L'' \) must get less of the private good at \( \Omega'' \) and since \( x_i \leq p(R_i^u) \) and \( x_i'' < x_i \) then \( x_i'' < p(R_i^u) \leq p(R_i^{\Omega''}) \) (where the last inequality may only be strict for agent \( f \)); moreover, for at least one such \( i \), it must be that \( x_l'' < x_i'' = x_f - \varepsilon \). Assume not; if for all agents \( i \) such that \( x_i'' < x_i \) we had \( x_i'' \geq x_f - \varepsilon \), then, given that \( x_i \leq x_f \), the sum of the increases \( x_i - x_i'' \) for all those agents could not exceed \(|N \setminus L''| \cdot \varepsilon \); but the sum of the increases in the private amount for all the agents in \( L'' = L \) is \(|L''| \cdot (x_f - \varepsilon - x_l) \). But then \(|L''| \cdot (x_f - \varepsilon - x_l) > |N \setminus L''| \cdot \varepsilon \) since \( \varepsilon < \frac{|I|}{|N|} (x_f - x_l) \) and \( \Omega'' = \Omega \) could not hold. Therefore, there is at least one agent \( i \) such that \( x_i'' < x_i'' \) and \( x_i'' < p(R_i^u) \). Clearly this is not a uniform distribution of \( \Omega'' \). Now, if there is an agent \( i \) such that \( p(R_i^{\Omega''}) \geq x_i'' > p(R_i^{\Omega''}) \) for all \( j \in L'' \), we go back to Step 1 in order to restart the accumulation at a higher private peak; again, there is a finite number of agents and this step can only be reached a finite number of times, at most until the highest private peak is reached as the accumulation point. Otherwise (Fig.13), we can proceed to Step 3.1. again (Fig.14).

Unlike the previous two steps, this one does not continue the accumulation process but it does allow it to continue as we proceed to Step 3.1. By
Figure 10: This is an illustration of the first substep of Step 3.3, following the case in Figure 5. Initially, $L = \{3, 5\}$ and $S \setminus L$ is empty. Agent 4 received the maximum private amount and is at his private peak. We then take a sequence $R'^v_5 \rightarrow R_4$, where $R^0_5 = R_5$ and $R'^1_5 = R_4$. Since the rule is weakly peak-only and weakly uncompromising, then throughout the sequence $x'^v_5 = x_5$ and $\Omega^v = \Omega$. At the limit, symmetry implies that the rule chooses $x^1_4 = x^1_5 = x_5$ for $\Omega^1 = \Omega$. Then, for continuity, we can find a preference relation in the sequence - that is represented by $p'_5$ in the figure - such that the rule chooses $\Omega' = \Omega$, $x'_4 = x_5$ and $x'_4 = x_4 - \varepsilon$ where $\varepsilon > 0$ is arbitrarily small.

Figure 11: This is an illustration of the second substep of Step 3.3, following the case in Figure 10. We now have $x_4 = p_4 - \varepsilon < p'_5$ (we use the notation $p'_5$ to emphasize the fact that these are not the initial accumulation preferences). We let agent 4’s preferences move from $R_4$ to $R'_5$. Since the rule is weakly uncompromising and weakly peak-only, the rule must still choose the same $x'_4 = x_4$ and $\Omega' = \Omega$ for $R'_5 = (R'_4, R'_5, R_{4-5})$. For symmetry to hold, we must also have $x'_4 = x'_5 > x_5$. But then since $\Omega$ must remain the same, the private amount of agent 5 increases, agent 4’s does not change, and $x_1, x_2$ and $x_3$ can at most decrease to 0, the sum of the private amounts cannot equal $\Omega$ and feasibility is violated.
successively manipulating preferences, we end up with the exact same accumulation set \( L'' = L \) at a higher private amount so that we can force the others to decrease as much as possible with the ultimate goal of achieving an incompatibility between efficiency and feasibility.

The need to return to Step 1 of the algorithm can only occur a finite number of times: it can either derive from the fact that an agent has a public peak different from \( \Omega \), but there is a finite number of times that this can happen since there is a finite number of agents and therefore a finite number of agents with public peaks different from \( \Omega \), and throughout the algorithm no agent with a public peak of \( \Omega \) changes his public peak to any other amount; or it can derive from the fact that at some stage an agent receives an amount that is not greater than his private peak but that is greater than the accumulation private peaks, but the accumulation process always involves one of the initial private peaks, it may only restart at a greater private peak, and there is a finite number of agents and therefore a finite number of private peaks; then, this can happen at most until the highest private peak is reached as the accumulation point and, therefore, a finite number of times. The finite number of agents ensures the finite repetition of Step 1.

Ignoring now the need to return to Step 1, the process is also finite. Once there is no need to restart the algorithm, and Step 2 has been applied, the next step is either Step 3.1, 3.2 or 3.3; but then, Steps 3.2 and 3.3 are always necessarily followed by Step 3.1 and Step 3.1 is always necessarily followed by either Step 3.1, 3.2 or 3.3 again. Each application of Steps 3.1 and 3.2 adds one agent to the accumulation set that was started at Step 2 and each application of Step 3.3 leaves the set of agents unchanged. Therefore, unless
Figure 13: This is an illustration of the third substep of Step 3.3. Following Figure 12, we can now move agent 5's preferences back to the original accumulation preferences. Since the rule is weakly uncompromising and weakly peak-only, it must choose for $R' = (R'_4 = R'_5, R_{-4})$ a $\Omega' = \Omega$ and $x'_5 = x_5 = p_4 - \varepsilon$, and all the other agents in the original accumulation set (just agent 3 in this case) must also get that private amount. If we had more agents in the original accumulation set, we might again reach a contradiction with feasibility. In this case, however, we now have that no agent starts receiving a private amount greater than $p_3$, $L' = \{3, 5\}$ and $S' \setminus L' = \{4\}$ is non-empty; moreover, agent 4 is in $S' \setminus L'$ and $x'_4 < x'_3 = x'_5$. We can then move to Step 3.1, beginning with the same accumulation set as in the beginning of Step 3.3.

Figure 14: Following the case in Figure 13, we proceed with the algorithm by reapplying Step 3.1. By moving agent 4 to the accumulation set, we know that his private amount must remain the same and all the other agents in the accumulation set must receive $x'_4$ as well, for symmetry. But then efficiency and feasibility are again incompatible.
a contradiction is reached along the way, a contradiction is reached at a final step (either 3.1 or 3.2) when only one agent is outside the accumulation set and is not receiving the accumulation point, given that the distribution was not the uniform; once we change that agent’s preferences so that we move him to the accumulation set, every agent must get the same amount that is equal to what that agent was getting before and it is then impossible to keep $\Omega$ constant.

Notice that throughout the algorithm we have been considering the case $\Omega \leq E$ and ignoring the additional efficiency requirement that if $\Omega < E$, there must be an agent $j \in N$ such that $x_j = p(R_i^x)$ and $p(R_i^{\Omega}) \leq \Omega$. If we were to add this requirement, it could only make it easier to reach a contradiction in Steps 2, 3.1, 3.2 and 3.3.

Therefore, a uniform distribution must be used in any rule that satisfies efficiency, strategy-proofness, continuity and symmetry.

We now focus on rules that use a uniform distribution - and for these rules, we actually have equivalence between the peak-only and uncompromising axioms and their weak counterparts:

**Lemma 4** A rule that uses a uniform distribution is weakly peak-only if and only if it is peak-only, and weakly uncompromising if and only if it is uncompromising.

**Proof.** The fact that each property implies its weaker version is trivial. Also if a rule is weakly peak-only, then, as agent $i$’s preferences change from $R_i$ to $R_i'$, where $p(R_i^x) = p(R_i'^x)$ and $p(R_i^{\Omega}) = p(R_i'^{\Omega})$, the rule chooses $(\Omega, x)$ for $R$ and $(\Omega', x')$ for $R'$ where $\Omega' = \Omega$ and $x' = x_i$. Given that a uniform distribution is only a function of $\Omega$ and the agents’ private peaks, $x' = U(\Omega') = U(\Omega) = x$. Changing recursively each agent’s preferences without changing that agent’s peaks therefore leads to the exact same choice by the rule throughout, and the rule is peak-only. Similarly, under the constraint to a uniform distribution, if a rule is weakly public-side uncompromising for the public component of an agent $i$’s bundle, it is also public-side uncompromising: if the same value for $\Omega$ is chosen, and the private peaks are the same, $x = U(\Omega)$ is also the same. Also, under the constraint to a uniform distribution, if a rule is weakly private-side uncompromising for both the public and private components of an agent $i$’s bundle, it is also private-side uncompromising: if the same value for $\Omega$ is chosen, and, without loss of generality, agent $i$ was getting an amount $\lambda < p(R_i^x)$ and now gets that same amount $\lambda \leq p(R_i'^x)$, and the private peaks of all other agents are the same, $U(\Omega)$ is also the same: every agent with a private peak greater than $\lambda$ still gets $\lambda$ and every agent with a private peak not greater than $\lambda$ still gets his private peak.
Similarly, the use of the uniform distribution also allows us to strengthen the closed range axiom. Once we constrain a rule to allocate the public good level according to a uniform distribution, that \textit{a priori} constraint associates with each level of the public good a unique level of the private good i.e. the imposition of a uniform distribution leads to a constrained schedule of pairs \((x_i, \Omega)\) where \(x_i = U_i(\Omega)\). Given that each agent has lexicographic preferences and cares first and foremost about the amount of the private good he is awarded, and since each level of \(\Omega\) is necessarily associated with a unique level of \(x_i\), the best possible value of \(\Omega\) for agent may no longer be his original public peak but rather the value of \(\Omega\) associated with the best \textit{pair} \((x_i^\ast, \Omega^\ast)\) in the constrained schedule (that is further constrained by the feasibility condition \(\Omega \leq E\)). The value of \(\Omega^\ast\) is his (uniform-)induced public peak.

**Definition 11**  Given a preference profile \(R \in \mathcal{R}_{sp, lex}^N\) and a social endowment \(E \in \mathbb{R}_+\), let \(\Omega \leq E\) and \(x = U(\Omega)\) define a subset of the original domain. Agent \(i\)'s \textit{(uniform-)induced public peak} \(p^U(R_i)^{16}\) is the level of the public good in the pair \((x_i^\ast, \Omega^\ast)\) that an agent prefers out of the schedule of pairs \((x_i, \Omega)\) in that subset.

It is easy to verify, using the uniform distribution case, that the induced public peak may or may not coincide with the original public peak. Finding the induced public peak is simply a lexicographic process: out of the constrained schedule, we look for the values of \(\Omega\) that lead him to receive a private good amount that is the closest to his private peak and then, out of those values for \(\Omega\) (if there is more than one), we pick the one that brings him the closest to his original public peak.

We can now incorporate these uniform-induced public peaks into a stronger version of closed range that also includes an additional invariance requirement in the choice of the public good level:

**Definition 12**  A rule has a \textit{closed uniform range} if for all \(i \in N\), for all \(R \in \mathcal{R}_{sp, lex}^N\).

\(^{16}\)If we wanted the notation for agent \(i\)'s uniform-induced public peak to fully represent the concept, we would need to use an expression such as \(p^U \mid_U (R_i, R_{\sim i})\) since this public peak is induced by a uniform distribution that is determined by the private-side preferences of all agents (and actually just the private peaks), and it also depends on the public-side preferences of agent \(i\) (and actually just the public peak). We chose a more economical notation \(p^U(R_i)\) that is meant to parallel the notation for the original private and public peaks, emphasizing however that this induced peak depends on both sides of agent \(i\)'s preference relation, while simultaneously identifying the agent whose induced public peak we are referring to. The reader should, nevertheless, keep in mind that this is an induced public peak and that it may also vary with the other agents’ private-side preferences, which is only implicit in the use of the superscript \(U\).
• if \( p(R_i^f) = 0 \), the rule chooses \( x_i \) for \( R \); if \( p(R_i^f) = E \), the rule chooses \( x_i \geq x_i^* \) for \( R \); and for any \( p(R_i^f) \), the rule chooses \( x_i = \text{med}\{p(R_i^f), x_i, \overline{x}_i\} \) for \( R \).

• if \( x_i > 0 \), for all \( x_i \in \text{bdy} [x_i, \overline{x}_i] \), if \( p(R_i^f) \) is such that the rule chooses \( x_i \) for \((R_i, R_{-i})\), if \( p(R_i^0) = 0 \), the rule chooses \( \underline{\Omega}(x_i) \) for \( R \); if \( p(R_i^f) = E \), the rule chooses \( \overline{\underline{\Omega}}(x_i) = \underline{\Omega}(x_i) \) for \( R \); and for any \( p(R_i^f) \), the rule chooses \( \Omega = \overline{\underline{\Omega}}(x_i) = \underline{\Omega}(x_i) = \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \) for \( R \);

Moreover, \( \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} = \text{med}\{p^U(R_i), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \).

• for all \( x_i \in (x_i, \overline{x}_i) \), and for \( x_i = \overline{x}_i \) if \( x_i = 0 \), if \( p(R_i^f) \) is such that the rule chooses \( x_i \) for \((R_i, R_{-i})\), if \( p(R_i^0) = 0 \), the rule chooses \( \underline{\Omega}(x_i) \) for \( R \); if \( p(R_i^f) = E \), the rule chooses \( \overline{\underline{\Omega}}(x_i) \geq \underline{\Omega}(x_i) \) for \( R \); and for any \( p(R_i^f) \), the rule chooses \( \Omega = \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \) for \( R \);

Moreover, \( \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} = \text{med}\{p^U(R_i), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \).

Again, we obtain an equivalence relation between this axiom and its weaker version for a rule that uses a uniform distribution.

**Lemma 5** A rule that uses a uniform distribution has a closed range if and only if it has a closed uniform range.

**Proof.** If a rule that uses a uniform distribution has a closed range, then we know that for all \( x_i \in \text{bdy} [x_i, \overline{x}_i] \), if \( p(R_i^f) \) is such that the rule chooses \( x_i \) for \((R_i, R_{-i})\), if \( p(R_i^0) = 0 \), the rule chooses \( \underline{\Omega}(x_i) \) for \( R \); if \( p(R_i^f) = E \), the rule chooses \( \overline{\underline{\Omega}}(x_i) \geq \underline{\Omega}(x_i) \) for \( R \); and for any \( p(R_i^0) \), the rule chooses \( \Omega = \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \) for \( R \). Let \( x_i = \overline{x}_i \); then we know that for all \( R_i \) such that \( p(R_i^f) \geq \overline{x}_i \), the rule gives agent \( i \) a private amount \( \overline{x}_i \) and chooses \( \Omega = \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \) where \( \overline{\underline{\Omega}}(x_i) \leq \overline{\Omega}(x_i) \); we only need to verify that \( \overline{\underline{\Omega}}(x_i) = \overline{\Omega}(x_i) \). Let agent \( i \)'s preferences be such that \( p(R_i^f) > x_i = x_i \). Since \( x = U(\Omega), x_i = \lambda < p(R_i^f) \) and any agent with a private peak greater than \( \lambda \) must get \( \lambda \) at \( \Omega \) and any agent with a private peak not greater than \( \lambda \) must get his private peak at \( \Omega \). Therefore, regardless of the public peak that the agent announces, the public good level is unequivocally determined by the value of \( \lambda \). Therefore, whether he announces a public peak of \( 0 \) or of \( E \), \( \Omega \) cannot change and \( \overline{\underline{\Omega}}(x_i) = \overline{\Omega}(x_i) \).

Therefore, if \( p(R_i^f) \) is such that the rule chooses \( x_i \) for \( R \), the rule also chooses \( \Omega = \overline{\underline{\Omega}}(x_i) = \overline{\Omega}(x_i) = \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \). Also, the fact that \( \text{med}\{p(R_i^0), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} = \text{med}\{p^U(R_i), \underline{\Omega}(x_i), \overline{\Omega}(x_i)\} \) is trivial.

A symmetric argument can be used for the case where \( x_i = \underline{x}_i \) since the requirement that \( x_i > 0 \) ensures that we can let agent \( i \)'s preferences be such that \( p(R_i^f) < x_i = x_i \).
Although we can check that the rule is strategy-proof and peak-only, it 
preferences change, the continuity of 
by imposing continuity is the following: 
that the rule chooses $x_i$ for $(R_i, R_{-i})$, if $p(R_i) = 0$, the rule chooses $\Omega_i(x_i)$ for $R$; if $p(R_i^c) = E$, the rule chooses $\overline{\Omega_i}(x_i) \geq \Omega_i(x_i)$ for $R$; and for any $p(R_i)$, the rule chooses $\Omega = \text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$ for $R$. Also, we have that $\text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\} = \text{med}\{p^U(R_i), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$: since the agent receives his private peak, if $p(R_i^c) > \overline{\Omega_i}(x_i)$, then $p^U(R_i) \geq \overline{\Omega_i}(x_i)$ and $\overline{\Omega_i}(x_i) = \text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\} = \text{med}\{p^U(R_i), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$; if $p(R_i^c) < \Omega_i(x_i)$, then $p^U(R_i) \leq \Omega_i(x_i)$ and $\Omega_i(x_i) = \text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$ and if $p(R_i^c) \in [\Omega_i(x_i), \overline{\Omega_i}(x_i)]$, then $p^U(R_i) = p(R_i) = \text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\} = \text{med}\{p^U(R_i), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$.

Therefore, if the rule that uses a uniform distribution has a closed range, it has a closed uniform range and the reverse implication is trivial.

We can therefore restate Proposition 1 for a rule that uses a uniform distribution: a strategy-proof and continuous rule that uses a uniform distribution is peak-only, uncompromising and has a closed uniform range. Also, a rule that uses a uniform distribution has a closed uniform range if and only if it is strategy-proof and peak-only or, equivalently, if and only if it is strategy-proof and uncompromising.

We know that a rule that uses a uniform distribution and has a closed uniform range must be strategy-proof but it does not, however, have to be continuous. We know that for all $i \in N$, for all $R \in R_{sp,lex}$, if $p(R_i^c) = 0$, the rule chooses $x_i$ for $R$; if $p(R_i^c) = E$, the rule chooses $\overline{x}_i \geq x_i$ for $R$; and for any $p(R_i^c)$, the rule chooses $x_i = \text{med}\{p(R_i^c), x_i, \overline{x}_i\}$ for $R$; therefore, the choice of $x_i$ is continuous with respect to changes in the agent’s preferences. We also know that for all $i \in N$, for all $R \in R_{sp,lex}$, for all $x_i \in [x_i, \overline{x}_i]$, if $p(R_i^c)$ is such that the rule chooses $x_i$ for $(R_i, R_{-i})$, if $p(R_i^c) = 0$, the rule chooses $\Omega_i(x_i)$ for $R$; if $p(R_i^c) = E$, the rule chooses $\overline{\Omega_i}(x_i) \geq \Omega_i(x_i)$ for $R$; and for any $p(R_i^c)$, the rule chooses $\Omega = \text{med}\{p(R_i^c), \Omega_i(x_i), \overline{\Omega_i}(x_i)\}$ for $R$; therefore, the choice of $\Omega$ is continuous with respect to changes in the agent’s public-side preferences; and under a uniform distribution, so is $x$, and, therefore, the rule. We also know from Lemma 5 that, under a uniform distribution, if $x_i \in \text{bdy}\{[x_i, \overline{x}_i]\}$ and the private-side preferences change such that the same $x_i$ is chosen, $\Omega$ must remain constant and the rule is also continuous with respect to this change in preferences. However, if $x_i = p(R_i^c) = \text{med}\{p(R_i^c), x_i, \overline{x}_i\} \in (x_i, \overline{x}_i)$, and the agent’s private-side preferences change, the continuity of $\Omega$ is not assured.

An example of a rule that represents the type we are trying to exclude by imposing continuity is the following: $[\Omega = \min_i \{p^U(R_i)\}, U(\Omega)]$ if there is an agent $i$ such that $p(R_i^c) < \alpha$ and $[\Omega = \sum_{i \in N} p(R_i^c), U(\Omega)]$ otherwise. Although we can check that the rule is strategy-proof and peak-only, it
fails to satisfy continuity. For the case where \( i \)'s preferences are \( R_i \) with \( p(R_i) = \alpha \), and for all \( j \neq i \), \( p(R_j) > \alpha \) and \( \min_{j \neq i} \{ p^U(R_j) \} < \sum_{i \in \mathcal{N}} p(R_i^c) \), every agent would get his private peak and for agent \( i \), \( \Omega_i(\alpha) = \overline{\Omega}_i(\alpha) = \sum_{i \in \mathcal{N}} p(R_i^c) \). However, if \( i \)'s preferences were \( R_i^c \) instead, where \( p(R_i^c) = \alpha - \varepsilon \), with \( \varepsilon \) arbitrarily small, \( \Omega \) would jump to \( \min \{ p^U(R_j) \} \). Notice that in this case, and letting \( R_i' \) be such that \( R_i' = R_i^c \) and \( p(R_i^c) = 0 \), we would now have \( \Omega_i(x_i = \alpha - \varepsilon) \leq p^U(R_i') \) and \( \overline{\Omega}_i(\alpha - \varepsilon) = \min_{j \neq i} \{ p^U(R_j) \} \).

Since, for each value of \( x_i \), we have a corresponding value for both \( \Omega_i(x_i) \) and \( \overline{\Omega}_i(x_i) \), we can explicitly define functions \( \Omega_i : [\underline{x}, \overline{x}_i] \rightarrow [0, E] \) and \( \overline{\Omega}_i : [\underline{x}_i, \overline{x}_i] \rightarrow [0, E] \). The continuity issue arises only when these functions vary with the agent’s private-side preferences and the corresponding private amount in a discrete manner, like in the example above. Given that the lexicographic preferences are not continuous, imposing continuity of the rule with respect to the preferences requires an additional condition:

**Lemma 6** If a continuous rule that uses a uniform distribution has a closed range, then for all \( i \in \mathcal{N} \), for all \( R_{i-1} \in \mathbb{R}_{sp.lex}^{N-1} \), \( \Omega_i \) and \( \overline{\Omega}_i \) are continuous.

**Proof.** Without loss of generality, assume that for some agent \( i \), given the other agents’ preferences, there is a discrete jump in \( \underline{\Omega}_i \) at \( x_i \in [\underline{x}, \overline{x}_i] \) i.e. we can find a sequence \( x_i^v \rightarrow x_i \) where \( x_i^v \in [\underline{x}, \overline{x}_i] \) and \( v \in [0, 1] \) such that \( \Omega_i(x_i^v) \rightarrow \Omega_i(x_i) \). Then, we can find preferences \( R_i = \{ R_i^c, R_i^\Omega \} \) where \( p(R_i^c) = x_i \) and \( p(R_i^\Omega) = 0 \) and a sequence \( R_i^v \rightarrow R_i \) such that \( p(R_i^v) = x_i^v \) and \( R_i^\Omega = R_i^\Omega \) for all such \( v \), and since the rule has a closed range, it must choose \( x_i^v \) and \( \Omega^v \) for \( (R_i^v, R_{i-1}) \) where \( \Omega^v = \Omega_i(x_i^v) \). At the limit, the rule must choose \( x_i \) and \( \Omega = \Omega_i(x_i) \) and since \( \Omega_i(x_i^v) \rightarrow \Omega_i(x_i) \), \( \Omega^v \rightarrow \Omega \) and the rule would not be continuous.

Following Côrte-Real (2001), letting \( \Omega^U \) denote the minimum efficient choice for \( \Omega \), given the restriction to a uniform distribution, \( \Omega^U \) is simply the minimum uniform-induced public peak \( \min \{ p^U(R_i) \} \); similarly, letting \( \overline{\Omega}^U \) denote the maximum efficient choice for \( \Omega \), given the same restriction, \( \overline{\Omega}^U = \max \{ p^U(R_i) \} \). (\( \Omega \in [\Omega^U, \overline{\Omega}^U], U(\Omega) \)) defines the subset of efficient allocations with a uniform distribution.

Before presenting the next theorem, we introduce another requirement on \( \Omega_i \) and \( \overline{\Omega}_i \), that is a consequence of imposing efficiency:

**Lemma 7** If an efficient rule that uses a uniform distribution has a closed range, then for all \( i \in \mathcal{N} \), for all \( R_{i-1} \in \mathbb{R}_{sp.lex}^{N-1} \), if \( p^U(R_i) > \Omega_i(x_i) \), then \( \overline{\Omega}_i(x_i) \geq \Omega^U \) and if \( p^U(R_i) < \Omega_i(x_i) \), then \( \Omega_i(x_i) \leq \Omega^U \).

**Proof.** We know from Lemma 5 that a rule that uses a uniform distribution and has a closed range also has a closed uniform range. Assume then that the
claim does not hold; without loss of generality, let $R$ be a preference profile such that for some agent $i$, $p^U(R_i) > \Omega_i(x_i)$ and $\Omega_i(x_i) < \Omega^U_i$. Since $\Omega = med \left\{ p^U(R_i), \Omega_i(x_i), \Omega_i(x_i) \right\}$, where $\Omega_i(x_i) \leq \Omega_i(x_i)$, then $\Omega = \Omega_i(x_i) < \Omega^U_i$, violating efficiency. $\blacksquare$

We can now establish the characteristics of any element of the set of rules that satisfy strategy-proofness, continuity, efficiency and symmetry:

**Theorem 2** A rule satisfies strategy-proofness, continuity, efficiency and symmetry if and only if it uses a uniform distribution, has a closed range where, for all $i \in N$, for all $R_{-i} \in \mathbb{R}_{\text{p.lex}}^{N-1}$, $\Omega_i$ and $\bar{\Omega}_i$ are continuous, and if $p^U(R_i) > \Omega_i(x_i)$, then $\Omega_i(x_i) \geq \Omega^U_i$ and if $p^U(R_i) < \Omega_i(x_i)$, then $\Omega_i(x_i) \leq \Omega^U_i$.

**Proof.** Necessity follows from Theorem 1, Proposition 1 and Lemmas 6 and 7. We only need to show sufficiency. Since the rule has a closed range, it is strategy-proof as we established in Proposition 1. Since the rule has a closed range, the choice of $\Omega$ is continuous with respect to the public-side preferences of each agent; and since $\Omega_i$ and $\bar{\Omega}_i$ are also continuous in an agent’s private good amount that is, in turn, continuous with respect to the private-side preferences of that agent, the choice of $\Omega$ is also continuous with respect to the private-side preferences of each agent; finally, since a uniform distribution is continuous with respect to $\Omega$ and the private peaks, continuity is satisfied. Since the rule uses a uniform distribution and has a closed range, we know from Lemma 5 that for each agent, we can write $\Omega = med \left\{ p^U(R_i), \Omega_i(x_i), \Omega_i(x_i) \right\}$ with $\Omega_i(x_i) \leq \Omega_i(x_i)$; therefore, if $p^U(R_i) \in [\Omega_i(x_i), \Omega_i(x_i)]$, $\Omega = p^U(R_i)$ and $\Omega \in \left[ \Omega^U_i, \Omega^U \right]$; if $p^U(R_i) \notin [\Omega_i(x_i), \Omega_i(x_i)]$, without loss of generality, let $p^U(R_i) > \Omega_i(x_i)$; then, $\Omega = \Omega_i(x_i) < p^U(R_i) \leq \Omega^U$ and we also have that $\Omega_i(x_i) \geq \Omega^U_i$, and $\Omega \in \left[ \Omega^U_i, \Omega^U \right]$; therefore, the rule always chooses $\Omega \in \left[ \Omega^U_i, \Omega^U \right]$ and we already know that this choice of $\Omega$ paired with its uniform distribution is efficient. Finally, since the rule uses a uniform distribution, symmetry is satisfied. $\blacksquare$

In Côrte-Real (2001), we established that $(\Omega^U, U(\Omega^U))$ is the unique resource-monotonic\footnote{A resource-monotonic rule is such that, as the social endowment increases, all agents have their welfares changed in the same direction. See Thomson (1999a) for a study of this property across several models.} selection from the set of efficient and envy-free allocations and that $(\min\left\{ \sum_{i \in N} p(R_i^f), E \right\}, U(\min\{ \sum_{i \in N} p(R_i^f), E \}))$ is the unique...
consistent\textsuperscript{18} and peak-only selection from the set of efficient and envy-free allocations. We can now determine whether each of these rules, which we know to be efficient, is strategy-proof.

We first consider the rule that picks \((\Omega^U, U(\Omega^U))\), i.e. the minimum uniform-induced public peak as the public good level, associated with its uniform distribution among the agents. We would like to determine if it is strategy-proof, but we can do that by looking at a more general rule instead - a rule we name the 'uniform generalized median-voter':

\begin{definition}
Given a preference profile \(R \in \mathbb{R}^{N}_{sp,lex}\) and a social endowment \(E \in \mathbb{R}^{+}\), the uniform generalized median-voter rule chooses a public good level equal to the median of a set that contains the \(N\) uniform-induced public peaks and \(N - 1\) additional constants, along with its uniform distribution:

\[
\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}, U(\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\})
\]

where each \(\alpha_i\) is a constant in \(\mathbb{R}^{+}\cup\{\infty\}\).
\end{definition}

It is straightforward to check that the rule that picks \((\Omega^U, U(\Omega^U))\) is a special case of the uniform generalized median-voter rule where \(\alpha_i = 0\) for all \(i = 1, \ldots, N - 1\).

This uniform generalized median-voter rule is a combination of the generalized median-voter rule from the public-good model and the uniform rule from the private-good model. We should however note that, unlike its public-good model counterpart, the uniform generalized median-voter rule does not consider the original public peaks in the choice of the public good level but rather the induced public peaks under the restriction to a uniform distribution, since these are now the relevant public peaks, due to the lexicographic preferences. We should also note that, in their respective settings, both the uniform rule and the generalized median-voter rule have been established as not only strategy-proof and efficient, but also as the only rules satisfying those properties together with continuity and an anonymity condition. Therefore, besides determining the implementability of the rule that chooses \((\Omega^U, U(\Omega^U))\), an analysis of the uniform generalized median-voter rule has the additional relevance of allowing us to determine how the results from the private- and public-good models extend to the mixed-good model.

We can start by checking that the uniform generalized median-voter rule is peak-only. A uniform distribution of a given public good level depends

\textsuperscript{18} A rule is consistent if the restriction of the desirable allocation selected for a set of agents \(N\) to any subgroup \(N' \subset N\) is still desirable for that subgroup. See Thomson (1999b) for a comprehensive formal analysis of consistency.
Proof. The uniform generalized median-voter rule, 
\[
\left(\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}, U(\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\})\},
\]
where each \(\alpha_i\) is a constant in \(\mathbb{R}_+ \cup \{\infty\}\), is strategy-proof, continuous, efficient and symmetric.

Proposition 2. The uniform generalized median-voter rule,
\( \Omega = \Omega' \) and \( x_i = x'_i = \xi_i \). Also, if \( \xi_i = \tilde{E} \) and \( R_i \) is such that \( p(R_i) = \xi_i \), then \( x_i < x_i \leq E \) cannot hold; otherwise, again from Step 1 we would need \( \xi_i = x_i < E \); therefore, \( x_i = \xi_i = \tilde{E} \) and \( \Omega = \Omega' = \tilde{E} \). Therefore, for all \( R_i \) such that \( p(R_i) \geq \xi_i \), \( \Omega = \Omega' \) and \( x_i = x'_i = \xi_i \).

In order to show that \( \xi_i \leq \xi_j \) we now assume towards a contradiction that \( 0 \leq \xi_i < \xi_j \leq E \); but then \( \xi_j < E \) and \( 0 \leq \xi_i \), and for \( R_i \) such that \( p(R_j) \in (\xi_i, \xi_j) \) we know from Step 1 that we must have both \( x'_i = \tilde{E} \) and \( x'_j = \tilde{E} \), which is impossible with \( \xi_i < \xi_j \). Therefore, \( \xi_i \geq \xi_j \).

In order to show that for all \( R_i \) such that \( p(R_i) \in [\xi_i, \xi_j] \), we have \( x_i = p(R_i) \), we now assume, without loss of generality, and towards a contradiction that \( p(R_i) \in [\xi_i, \xi_j] \) and \( x_i < p(R_i) \). But then since for \( R_i \) such that \( p(R_i) = \tilde{E} \), \( x'_i = \tilde{E} \), and since \( p(R_i) \geq p(R_i) > x_i \), we know from Step 1 that we must have \( x'_i = x_i < p(R_i) \leq \xi_i \) which contradicts \( x'_i = \xi_i \). Therefore, for all \( R_i \) such that \( p(R_i) \in [\xi_i, \xi_j] \), we have \( x_i = p(R_i) \).

We have established that if \( p(R_i) = 0 \), the rule chooses \( \xi_j \) for \( R_i \); if \( p(R_i) = \tilde{E} \), the rule chooses \( \xi_j \) for \( R_i \); and for any \( p(R_i) \), the rule chooses \( x_i = \text{med} \{ p(R_i), \xi_j, \xi_i \} \) for \( R_i \). Moreover, if \( \xi_j > 0 \), for all \( x_i \in \text{bdy}[\xi_i, \xi_j] \), if \( p(R_i) \) is such that \( x_i \) for \( (R_i, R_{-i}) \), the choice of \( \Omega \) remains the same i.e. \( \Omega \equiv \Omega_i(x_i) = \Omega_j(x_i) = \text{med} \{ p(R_i), \Omega_i(x_i), \Omega_i(x_i) \} = \text{med} \{ p(R_i), \Omega_i(x_i), \Omega_i(x_i) \} \).

Step 3: We only need to prove that for all \( x_i \in (\xi_j, \xi_i) \) whenever \( p(R_i) \) is such that the rule chooses \( x_i \) for \( (R_i, R_{-i}) \), if \( p(R_i) = 0 \), the rule chooses \( \Omega_j(x_i) \) for \( R_i \); if \( p(R_i) = \tilde{E} \), the rule chooses \( \Omega_j(x_i) \) for \( R_i \); and for any \( p(R_i) \), the rule chooses \( \Omega = \text{med} \{ p(R_i), \Omega_j(x_i), \Omega_j(x_i) \} = \text{med} \{ p(R_i), \Omega_j(x_i), \Omega_j(x_i) \} \) for \( R_i \).

Since every agent’s uniform-induced public peak depends only on the private peaks of all agents and his public peak, a change in agent \( i \)’s public peak will only affect his induced public peak. Let \( R_i \) and \( \tilde{R}_i \) be such that \( p(R_i) = \tilde{R}_i = p(R_i) \) with \( p(R_i) = 0 \) and \( p(R_i) = \tilde{E} \) and let \( \Omega_j(x_i) \equiv \text{med} \{ p(R_i), p(R_i), p(R_i), ..., p(R_i), \alpha_1, ..., \alpha_{N-1} \} \) and \( \Omega_i(x_i) \equiv \text{med} \{ p(R_i), p(R_i), p(R_i), ..., p(R_i), \alpha_1, ..., \alpha_{N-1} \} \); since \( p(R_i) \leq p(R_i) \), \( \Omega_i(x_i) \leq \tilde{\Omega}_i(x_i) \).

Out of the set \( \{ p(R_i), ..., p(R_i), p(R_i), p(R_i), ..., p(R_i), \alpha_1, ..., \alpha_{N-1} \} \), \( N - 1 \) points are therefore smaller or equal to both \( \Omega_j(x_i) \) and \( \tilde{\Omega}_i(x_i) \) and \( N - 1 \) points are greater or equal to both \( \Omega_j(x_i) \) and \( \tilde{\Omega}_i(x_i) \). Therefore, since \( p(R_i) \in [p(R_i), p(R_i)] \) and \( \text{med} \{ p(R_i), p(R_i), p(R_i), p(R_i), p(R_i), \alpha_1, ..., \alpha_{N-1} \} \) will be \( \Omega_j(x_i) \) if \( p(R_i) \leq \Omega_j(x_i), \Omega_j(x_i) \) if \( p(R_i) \geq \Omega_j(x_i) \) and \( p(R_i) \) if \( p(R_i) \in [\Omega_j(x_i), \Omega_j(x_i)] \).

Therefore, \( \Omega = \text{med} \{ p(R_i), \Omega_j(x_i), \Omega_j(x_i) \} = \text{med} \{ p(R_i), \Omega_j(x_i), \Omega_j(x_i) \} \) and the rule has closed range.

The continuity of \( \Omega \) and \( \Omega \) is straightforward, as well as the requirements that \( \Omega_j(x_i) \geq \Omega \) and \( \Omega_j(x_i) \leq \tilde{\Omega} \).
The uniform generalized median-voter rule encompasses not only the rule that picks the lowest induced public peak, which is achieved by setting \( \alpha_i = 0 \) for all \( i = 1, \ldots, N - 1 \), but also the rule that selects the highest induced public peak, which is achieved by setting \( \alpha_i = \infty \) for all \( i = 1, \ldots, N - 1 \) and any rule that picks the \( n \)th lowest induced public peak (where \( N > n > 1 \)), which is achieved by setting \( \alpha_i = 0 \) for all \( i = 1, \ldots, N - n + 1, \ldots, N - 1 \). However, the uniform generalized median-voter rule does not include the rule that, given a preference profile \( R \in \mathbb{R}^N_{sp,lex} \) and a social endowment \( E \in \mathbb{R}_+ \), always chooses the sum of the private peaks and gives each agent his private peak, unless this is not feasible, in which case it uniformly distributes the social endowment i.e. the rule that picks \( \min \{ \sum_{i \in N} p(R_i^p), E \} \), \( U(\min \{ \sum_{i \in N} p(R_i^p), E \}) \).

Given that in Côrte-Real (2001) we also established its desirability from a normative standpoint, we are also interested in its implementability. When \( \sum_{i \in N} p(R_i^p) \) is feasible, there is always at least one induced public peak that is not greater than \( \sum_{i \in N} p(R_i^p) \) - at least one agent does not get his private peak at \( \Omega > \sum_{i \in N} p(R_i^p) \) - and always one induced public peak that is not smaller than \( \sum_{i \in N} p(R_i^p) \) - at least one agent does not get his private peak at \( \Omega < \sum_{i \in N} p(R_i^p) \) - and, therefore, if \( \sum_{i \in N} p(R_i^p) \) is feasible, it is in the range defined by the induced public peaks i.e. \( \sum_{i \in N} p(R_i^p) \in [\min_i \{ p^U(R_i) \}, \max_i \{ p^U(R_i) \}] \). Despite this fact, in order to ensure that \( \sum_{i \in N} p(R_i^p) \) is chosen whenever feasible, we would need a rule \( \{ \text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}, U(\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}) \} \), where each \( \alpha_i \) is no longer a constant in \( \mathbb{R}_+ \cup \{ +\infty \} \), but rather \( \alpha_i = \sum_{i \in N} p(R_i^p) \) for all \( i = 1, \ldots, N - 1 \) i.e. the parameters \( \alpha_i \) are now a specific function of the preferences and actually only of the private peaks. In order to present a useful fact about this rule, we need to introduce the formal definition of private-peak-only:

**Definition 14** A rule is private-peak-only if, for all \( R, R' \in \mathbb{R}^N_{sp,lex} \) such that, for all \( i \in N \), \( p(R_i^p) = p(R_i'^p) \), it chooses the same \( (\Omega, x) \) for \( R \) and \( R' \).

The rule that, given a preference profile \( R \in \mathbb{R}^N_{sp,lex} \) and a social endowment \( E \in \mathbb{R}_+ \), chooses \( \min \{ \sum_{i \in N} p(R_i^p), E \}, U(\min \{ \sum_{i \in N} p(R_i^p), E \}) \) is private-peak-only: we already know that a uniform distribution is determined solely by the agents’ private-peaks (and by the available amount that is to be distributed i.e. the public good level) and this rule is such that the choice of the public good level depends only on the private peaks as well. We can use this fact to apply Theorem 2 once more and prove the following result:

**Proposition 3** The rule that, given a preference profile \( R \in \mathbb{R}^N_{sp,lex} \) and a social endowment \( E \in \mathbb{R}_+ \), chooses the sum of the private peaks and gives
each agent his private peak, unless this is not feasible, in which case it uniformly distributes the social endowment, i.e. the rule that picks \( \min \{ \sum_{i \in N} p(R_i^x), E \} \), is strategy-proof, continuous, efficient and symmetric.

**Proof.** Following Theorem 2, we must prove that this rule that uses a uniform distribution has a closed range. Let \( (\Omega, x) \) be chosen by the rule for the economy with the preference profile \( R \). Note that the rule is such that \( \Omega = \sum_{i \in N} p(R_i^x) \) and \( U_i(\Omega) = p(R_i^x) \) for all \( i \in N \). Let \( R_i' \) be an alternative preference relation for agent \( i \) and let \( (\Omega', x') \) be chosen by the rule for the economy with the preference profile \( (R_i', R_{-i}) \). If \( R_i' \) is such that \( p(R_i^x') = 0 \), and since \( \Omega' \leq \sum_{j \neq i} p(R_j^x) \) and \( x_i' = U_i(\Omega) \leq p(R_i^x) \), we must have \( x_i' = 0 \) and \( x_j \equiv x_j' = 0 \).

Now let \( R_i' \) be such that \( p(R_i^x') = E \). Since \( \Omega' = E, x_i' = U_i'(E) \leq p(R_i^x') \), for all \( j : p(R_j^x) \geq x_i' \) we have \( U_j'(E) = x_i' \) and \( \Omega' = \sum_{j \neq i} p(R_j^x) + x_i' + \sum_{j \neq i : p(R_j^x) < x_i'} p(R_j^x) + x_i' = p(R_i^x') + x_i' + \sum_{j \neq i} x_i' = p(R_i^x') + x_i' = E \). Let \( \pi_i \equiv x_i' \). Now if \( p(R_i^x) > \pi_i \), we have \( \sum_{i \in N} p(R_i^x) \geq \pi_i \geq \pi_i \) and therefore \( \Omega = E \) and \( x_i = \pi_i = U_i'(E) = U_j'(E) \) for all \( j : p(R_j^x) \geq \pi_i \). Clearly, \( \pi_i \geq x_i \) if \( p(R_i^x) \leq \pi_i \) and \( x_i = p(R_i^x) \). Otherwise, we would have \( x_i < p(R_i^x) \) and \( \Omega = \sum_{j \in p(R_j^x) < x_i} p(R_j^x) + x_i' + \sum_{j \neq i} x_i' \). Now if \( p(R_i^x) < \pi_i \), \( x_i = p(R_i^x) \) and \( (\Omega, x) \) is such that the rule chooses \( \pi_i \) for \( R \). If \( p(R_i^x) = \pi_i \), the rule chooses \( \pi_i \) for \( R \); and for any \( p(R_i^x) \), the rule chooses \( x_i = \text{med} \{ p(R_i^x), \pi_i, \pi_i \} \) for \( R \). Moreover, since the rule is private-peak-only, changing an agent’s public peak will not influence the choice of the rule and for all \( x_i \in [\pi_i, \pi_i] \), if \( p(R_i^x) \) is such that the rule chooses \( x_i \) for \( (R_i, R_{-i}) \), the choice of \( \Omega \) remains the same i.e. \( \Omega = \pi_i \).

Therefore, the rule has closed range. The continuity of \( \Omega \) and \( \Omega' \) is straightforward, and since \( \sum_{i \in N} p(R_i^x) \in \{ \min_i \{ p(R_i(E)) \}, \max_i \{ p(R_i(E)) \} \} \), so are the requirements that \( \Omega_i(x_i) \geq \Omega_i^U \) and \( \Omega_i(x_i) \leq \Omega_i^U \).

**Proposition 2** tells us that, like the uniform rule in the private-good case and the generalized median-voter rule in the public-good case, the uniform generalized median-voter rule is efficient and strategy-proof. Proposition 3, however, tells us that, whereas in each of those one-dimensional models those rules were also the only rules satisfying both properties together with continuity and an anonymity condition, the uniform generalized median-voter is not the unique rule satisfying those same properties in the mixed-good model.

Instead, the same requirements applied to the mixed-good model allow for more freedom in the way the rule chooses the public good level.
Rather than requiring that each $\alpha_i$ be a constant in $\mathbb{R}_+ \cup \{\infty\}$ in the rule 
$\left( \text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}, U(\text{med}\{p^U(R_1), \ldots, p^U(R_N), \alpha_1, \ldots, \alpha_{N-1}\}) \right)$, we have to allow for each $\alpha_i$ to vary with the preferences as well in order to obtain other rules in the set described in Theorem 2, such as the rule studied in Proposition 3.

Given our restriction to lexicographic preferences, this result is perhaps not surprising: the fact that the public good is a subsidiary concern for the agents means that the choice of the public good level is less relevant than in the pure public-good model, and therefore the imposition of strategy-proofness is also less demanding on that choice. In any case, our mixed-good model does enable us to assert the need for the use of a uniform distribution in any allocation rule, and that need is justified by both normative and strategic concerns. In that respect, the analysis of the mixed-good model reinforces the results obtained for the uniform rule in private-good models.

4 Conclusions

We conduct a strategic analysis of a model of a mixed-good economy. We decompose the mixed good into a private and a public good and assume that the agents have separably single-peaked and lexicographic preferences, where the primary concern is the private good.

We first try to extract the implications of strategy-proofness for a rule in this model and, adding the requirement of continuity, conclude that such a rule must be weakly uncompromising, weakly peak-only and have a closed range. These conclusions help us establish a fundamental result: that a uniform distribution of the public good level must be used in any strategy-proof, continuous and efficient rule that also satisfies symmetry. We finally extract all the implications of this set of properties and conclude that even though we must have a unique distribution of the public good level, we have more flexibility in the choice of that level, and that is a consequence of the restriction to lexicographic preferences where the private good is the primary concern. Rather than a unique rule, we therefore obtain an equivalence between that set of properties and a set of characteristics that the rule must exhibit: besides the aforementioned use of a uniform distribution, it must have a closed range and satisfy two additional requirements that ensure a continuous and efficient choice of the public good level.

We then proceed to use this characterization in the analysis of the uniform generalized median-voter rule, that encompasses the rule we characterized in Côrte-Real (2001) as the unique one satisfying efficiency, no-envy and resource-monotonicity. We conclude that it is strategy-proof, continuous, efficient and symmetric but whereas each of its counterparts in the private-
and the public-good models (respectively, the uniform rule and the general-
ized median-voter rule) can be characterized by that set of properties, the
uniform generalized median-voter rule is not the unique rule that satisfies
them in our setting. Namely, another rule fulfilling those requirements is
the one we characterized in Côrte-Real (2001) as the unique rule satisfying
efficiency, no-envy, consistency and peak-only. These results also allow us
to establish that, when we try to implement each of the rules for which we
found characterizations in Côrte-Real (2001) based on theoretical proper-
ties that we deemed appealing from a normative viewpoint, those properties
will indeed be satisfied, given that no individual manipulation of the rule is
profitable and therefore none will be attempted.

Since our characterization of a rule that satisfies strategy-proofness, con-
tinuity, efficiency and symmetry is based on the associated manipulation
sets i.e. on the range of the rule with all but one component of the economy
held constant, finding a closed functional form for such a rule is a possibly
interesting direction for further research - that could also involve changes in
the model, like the ones suggested in Côrte-Real (2001).

Unlike other multidimensional models, our mixed-good model does not
lead to a necessary trade-off between strategy-proofness and efficiency. More-
over, these properties are also compatible with equity-based properties such
as no-envy. Both strategic and normative concerns lead us to a clear recom-
mendation: the use of a uniform distribution, regardless of the associated
choice for the public good level.

References

R u l e s ', Economics Letters 40 (1992), 57-60.

Social Choice Functions for Economies with Pure Public Goods', Social

and Phantom Voters', The Review of Economic Studies 50 (1983), 153-
170.


[5] Ching, S., 'Strategy-Proofness and ”Median Voters”', International Jour-


