Crashes, Volatility, and the Equity Premium: Lessons from S&P500 Options

Appendix

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Stock Market Risk Premium

First, substitute (11) into (10) and use the fact that in equilibrium $w = 1$ to get the following PDE satisfied by the value function, J:

$$\begin{aligned} 0 &= J_t + rWJ_W - \frac{1}{2}W^2 J_{WW} Y^2 - Z^2 W \mathrm{E}_Q\left[J_W(W(1+Q), Y, Z, t)Q\right] + J_Y\left(\mu_Y + \kappa_Y Y\right) \\ &\quad + \frac{1}{2}J_{YY}\sigma_Y^2 + J_Z\left(\mu_Z + \kappa_Z Z\right) + \frac{1}{2}J_{ZZ}\sigma_Z^2 + J_{YZ}\rho_{YZ}\sigma_Y\sigma_Z + Z^2\mathrm{E}_Q\left[\Delta J\right]. \end{aligned} \tag{A.1}$$

In general there is no analytical solutions to this PDE. However, in the case of power utility function we can find one. Next, guess a solution of the following form:

$$J(W, Y, Z, t) = e^{r(1-\gamma)\tau} g(Y, Z, \tau)\frac{W^{1-\gamma}}{1-\gamma}, \tag{A.2}$$

where $g(Y, Z, \tau)$ is a function independent of W. Substituting (A.2) into (A.1) to get:

$$\begin{aligned} g_\tau &= \left(-\frac{1}{2}\gamma(\gamma-1)Y^2 + Z^2\mathrm{E}_Q\left[(1+\gamma Q)(1+Q)^{-\gamma} - 1\right]\right) g + \left(\mu_Y + \kappa_Y Y\right) g_Y \\ &\quad + \frac{1}{2}\sigma_Y^2 g_{YY} + \left(\mu_Z + \kappa_Z Z\right) g_Z + \frac{1}{2}\sigma_Z^2 g_{ZZ} + \rho_{YZ}\sigma_Y\sigma_Z g_{YZ}, \end{aligned} \tag{A.3}$$

with the initial condition $g(Y, Z, 0) = 1$. (A.3) is a hyperbolic PDE whose coefficients are quadratic functions of Y and Z. Again we make a guess of g of the form:[1]

$$g(Y, Z, \tau) = e^{A(\tau) + B(\tau)^\top U + U^\top C(\tau) U}, \tag{A.4}$$

where we define $U \equiv \left(\begin{smallmatrix} Y \\ Z \end{smallmatrix}\right)$, and $A(\tau)$ is a function with initial condition $A(0) = 0$. Substitute (A.4) into (A.3) and collect terms with the same powers of Y and Z. Since (A.3) holds for

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[1] This trick has been frequently used. See for example Ingersoll (1987) and Heston (1993).

any values of Y and Z, all the coefficients must equal zero. And this leads to the system of ODEs (14)-(15) together with:

$$A' = \Pi^\top B + \frac{1}{2} B^\top \Gamma B + \mathrm{tr}(\Gamma C). \tag{A.5}$$

Equation (13) is then obtained from (11), (A.2), and (A.4).

Option Pricing

Under the risk-adjusted probability measure the price, f, of a European call option with strike price K and maturity date T is a function of the state variables and time, (S, Y, Z^*, t). Letting the subscripts of f represent partial derivatives, then $f(S, Y, Z^*, t)$ satisfies the following PDE:[2]

$$\begin{aligned} -f_t \;=\; & -rf + \left(r - Z^{*2}\mu_Q^*\right) S f_S + \frac{1}{2} Y^2 S^2 f_{SS} + \left(\mu_Y^* + \kappa_{YY}^* Y + \kappa_{YZ}^* Z^*\right) f_Y + \frac{1}{2}\sigma_Y^2 f_{YY} \\ & + \left(\mu_Z^* + \kappa_{ZY}^* Y + \kappa_{ZZ}^* Z^*\right) f_{Z^*} + \frac{1}{2}\sigma_Z^{*2} f_{Z^*Z^*} + \rho_{SY}\sigma_Y Y S f_{SY} + \rho_{SZ}\sigma_Z^* Y S f_{SZ^*} \\ & + \rho_{YZ}\sigma_Y\sigma_Z^* f_{YZ^*} + Z^{*2}\mathrm{E}_{Q^*}\left[f(S(1+Q^*), Y, Z^*, t) - f(S, Y, Z^*, t)\right], \end{aligned} \tag{A.6}$$

where E_{Q^*} is the expectation with respect to the distribution of Q^* and the boundary condition is:

$$f(S, Y, Z^*, T) = (S - K)^+. \tag{A.7}$$

To simplify, define $x \equiv \ln S$. Then $f(x, Y, Z^*, T)$ satisfies:

$$\begin{aligned} -f_t \;=\; & -rf + \left(r - Z^{*2}\mu_Q^* - \frac{1}{2}Y^2\right) f_x + \frac{1}{2} Y^2 f_{xx} + \left(\mu_Y^* + \kappa_{YY}^* Y + \kappa_{YZ}^* Z^*\right) f_Y + \frac{1}{2}\sigma_Y^2 f_{YY} \\ & + \left(\mu_Z^* + \kappa_{ZY}^* Y + \kappa_{ZZ}^* Z^*\right) f_{Z^*} + \frac{1}{2}\sigma_Z^{*2} f_{Z^*Z^*} + \rho_{SY}\sigma_Y Y f_{xY} + \rho_{SZ}\sigma_Z^* Y f_{xZ^*} \\ & + \rho_{YZ}\sigma_Y\sigma_Z^* f_{YZ^*} + Z^{*2}\mathrm{E}_{Q^*}\left[f(x + \ln(1+Q^*), Y, Z^*, t) - f(x, Y, Z^*, t)\right]. \end{aligned} \tag{A.8}$$

We now use the Fourier transform of f to further simplify the above equation.[3] Let

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[2] A version of Ito's lemma for jump-diffusions is used in deriving the PDE. See for example Protter (1990).

[3] This technique is used in Heston (1993), Bates (1996), and Duffie, Pan, and Singleton (2000) among others. The pricing formulas derived in these papers generally involve two integrals. Our approach here is similar to Lewis (2000) in that we only need one integral.

$\hat{f}(k, Y, Z^*, t)$ be the Fourier transform of f with respect to x, that is:

$$\hat{f}(k, Y, Z^*, t) \equiv \int_{-\infty}^{\infty} e^{ikx} f(x, Y, Z^*, t) dx. \tag{A.9}$$

The boundary condition $f(x, Y, Z^*, T) = (e^x - K)^+$ changes to:

$$\hat{f}(k, Y, Z^*, T) = -\frac{K^{ik+1}}{k^2 - ik}.$$

If we write $k = k_r + ik_i$ where k_r and k_i are the real and imaginary parts of k respectively, then f is recovered via the inverse Fourier transform:

$$f(x, Y, Z^*, t) = \frac{1}{2\pi} \int_{ik_i - \infty}^{ik_i + \infty} e^{-ikx} \hat{f}(k, Y, Z^*, t) dk. \tag{A.10}$$

Differentiating (A.9), integrating by parts, and changing the order of expectation and Fourier transform in the last term, (A.8) becomes:

$$\begin{aligned} -\hat{f}_t &= -(1 + ik)r\hat{f} + ikZ^{*2}\mu_Q^*\hat{f} - \frac{1}{2}Y^2(k^2 - ik)\hat{f} + \left(\mu_Y^* + \kappa_{YY}^* Y - ik\rho_{SY}\sigma_Y Y + \kappa_{YZ}^* Z^*\right)\hat{f}_Y \\ &\quad + \frac{1}{2}\sigma_Y^2 \hat{f}_{YY} + \left(\mu_Z^* + \kappa_{ZY}^* Y - ik\rho_{SZ}\sigma_Z^* Y + \kappa_{ZZ}^* Z^*\right)\hat{f}_{Z^*} + \frac{1}{2}{\sigma_Z^*}^2 \hat{f}_{Z^* Z^*} \\ &\quad + \rho_{YZ}\sigma_Y\sigma_Z^*\hat{f}_{YZ^*} + Z^{*2}\mathrm{E}_{Q^*}\left[(1 + Q^*)^{-ik} - 1\right]\hat{f}. \end{aligned} \tag{A.11}$$

Notice that the jump variable Q^* is now separated from $\hat{f}$. If we define:

$$h \equiv e^{(1+ik)r\tau}\hat{f},$$

then (A.11) changes to:

$$\begin{aligned} -h_t &= \left[-\frac{1}{2}Y^2(k^2 - ik) + ikZ^{*2}\mu_Q^*\right] h + \left(\mu_Y^* + \kappa_{YY}^* Y - ik\rho_{SY}\sigma_Y Y + \kappa_{YZ}^* Z^*\right) h_Y \\ &\quad + \frac{1}{2}\sigma_Y^2 h_{YY} + \left(\mu_Z^* + \kappa_{ZY}^* Y - ik\rho_{SZ}\sigma_Z^* Y + \kappa_{ZZ}^* Z^*\right) h_{Z^*} + \frac{1}{2}{\sigma_Z^*}^2 h_{Z^* Z^*} \\ &\quad + \rho_{YZ}\sigma_Y\sigma_Z^* h_{YZ^*} + Z^{*2}\mathrm{E}_{Q^*}\left[(1 + Q^*)^{-ik} - 1\right] h, \end{aligned} \tag{A.12}$$

with the initial condition:

$$h(k, Y, Z^*, 0) = -\frac{K^{ik+1}}{k^2 - ik}. \tag{A.13}$$

To solve (A.12) with the initial condition (A.13), it is enough to solve the same equation with the initial value equal to one and then scale the solution by the r.h.s. of (A.13). Given the solution to (A.12) with the initial condition $h(k, Y, Z^*, 0) = 1$, the option price is:[4]

$$f(S, Y, Z^*, \tau) = S - \frac{e^{-r\tau}}{2\pi} \int_{\frac{i}{2}-\infty}^{\frac{i}{2}+\infty} e^{-ik(r\tau + \ln S)} \frac{K^{ik+1}}{k^2 - ik} h(k, Y, Z^*, \tau) dk. \tag{A.14}$$

Now the problem is to find a solution to (A.12) with initial value of one. Recognizing the similarity between (A.3) and (A.12), we use the same trick by guessing a solution as:

$$h(Y, Z^*, \tau) = e^{A^*(\tau) + B^*(\tau)^\top U^* + U^{*\top} C^*(\tau) U^*}. \tag{A.15}$$

Then by a similar calculation as before we can derive the system of ODEs (29)-(31).

Relation Between Probability Measures

Assume that, under the objective probability measure, the option price, $f(S, Y, Z, t)$, follows the process:

$$df = \left(r + \phi_f - \lambda \mu_{Q_f}\right) f dt + \sigma_{f_S} f dW_S + \sigma_{f_Y} f dW_Y + \sigma_{f_Z} f dW_Z + Q_f f dN, \tag{A.16}$$

where $Q_f \equiv \left[f(S(1+Q), Y, Z) - f(S, Y, Z)\right]/f$ is the percentage jump in the option price and μ_{Q_f} is the average jump size. ϕ_f is the risk premium on the option.

In the presence of the option market, the representative investor allocates his wealth in the stock, the option, and the risk free asset with the portfolio weights denoted by $(w, w_f, 1 - w - w_f)$. Investor's wealth, W, then follows the process:

$$\begin{aligned} dW &= (r + w\phi + w_f\phi_f - \lambda\mu_{Q_W}) W dt + wWY dW_S \\ &\quad + w_f\sigma_{f_S} W dW_S + w_f\sigma_{f_Y} W dW_Y + w_f\sigma_{f_Z} W dW_Z + Q_W W dN, \end{aligned} \tag{A.17}$$

where $Q_W = wQ + w_f Q_f$ is the percentage jump in wealth and $\mu_{Q_W} = w\mu_Q + w_f\mu_{Q_f}$ is the

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[4] This can be shown by using the inverse Fourier transform and the Residue theorem as in Lewis (2000).

average jump size in wealth. The value function J now solves the Bellman equation:

$$0 = \max_{w,w_f}\left[J_t + \mathcal{A}(J)\right], \tag{A.18}$$

with:

$$\begin{aligned}
\mathcal{A}(J) \;=\;& W J_W\left(r + w\phi + w_f\phi_f - Z^2\mu_{Q_W}\right) + J_Y\left(\mu_Y + \kappa_Y Y\right) + J_Z\left(\mu_Z + \kappa_Z Z\right)\\
&+\frac{1}{2}W^2 J_{WW}\Big[w^2Y^2 + 2ww_fY\left(\sigma_{f_S} + \rho_{SY}\sigma_{f_Y} + \rho_{SZ}\sigma_{f_Z}\right) + w_f^2\left(\sigma_{f_S}^2 + \sigma_{f_Y}^2 + \sigma_{f_Z}^2\right)\\
&+2w_f^2\left(\rho_{SY}\sigma_{f_S}\sigma_{f_Y} + \rho_{SZ}\sigma_{f_S}\sigma_{f_Z} + \rho_{YZ}\sigma_{f_Y}\sigma_{f_Z}\right)\Big] + \frac{1}{2}J_{YY}{\sigma_Y}^2 + \frac{1}{2}J_{ZZ}{\sigma_Z}^2\\
&+W J_{WY}\sigma_Y\left[w\rho_{SY}Y + w_f\left(\rho_{SY}\sigma_{f_S} + \sigma_{f_Y} + \rho_{SZ}\sigma_{f_Z}\right)\right]\\
&+W J_{WZ}\sigma_Z\left[w\rho_{SZ}Y + w_f\left(\rho_{SZ}\sigma_{f_S} + \rho_{YZ}\sigma_{f_Y} + \sigma_{f_Z}\right)\right]\\
&+J_{YZ}\rho_{YZ}\sigma_Y\sigma_Z + Z^2\mathrm{E}_{Q_W}\left[J(W(1+Q_W),Y,Z,t) - J(W,Y,Z,t)\right].
\end{aligned} \tag{A.19}$$

Differentiating (A.18) with respect to w_f and substitute in $w = 1, w_f = 0$, we obtain the risk premium on the option:

$$\begin{aligned}
\phi_f \;=\;& -W\frac{J_{WW}}{J_W}Y\left(\sigma_{f_S} + \rho_{SY}\sigma_{f_Y} + \rho_{SZ}\sigma_{f_Z}\right) - \frac{J_{WY}}{J_W}\sigma_Y\left(\rho_{SY}\sigma_{f_S} + \sigma_{f_Y} + \rho_{SZ}\sigma_{f_Z}\right)\\
&-\frac{J_{WZ}}{J_W}\sigma_Z\left(\rho_{SZ}\sigma_{f_S} + \rho_{YZ}\sigma_{f_Y} + \sigma_{f_Z}\right) - Z^2\mathrm{E}_Q\left[\frac{\Delta J_W}{J_W}Q_f\right].
\end{aligned} \tag{A.20}$$

On the other hand, by Ito's lemma the drift and the diffusion terms of df are:

$$\begin{aligned}
\phi_f f \;=\;& -rf + f_t + \left(r + \phi - Z^2\mu_Q\right)Sf_S + \left(\mu_Y + \kappa_Y Y\right)f_Y + \left(\mu_Z + \kappa_Z Z\right)f_Z\\
&+\frac{1}{2}Y^2S^2f_{SS} + \frac{1}{2}\sigma_Y^2 f_{YY} + \frac{1}{2}\sigma_Z^2 f_{ZZ} + \rho_{SY}\sigma_Y YSf_{SY} + \rho_{SZ}\sigma_Z YSf_{SZ}\\
&+\rho_{YZ}\sigma_Y\sigma_Z f_{YZ} + Z^2\mathrm{E}_Q\left[f(S(1+Q),Y,Z,t) - f(S,Y,Z,t)\right],
\end{aligned} \tag{A.21}$$

$$\sigma_{f_S} = SYf_S/f, \tag{A.22}$$

$$\sigma_{f_Y} = \sigma_Y f_Y/f, \tag{A.23}$$

$$\sigma_{f_Z} = \sigma_Z f_Z/f. \tag{A.24}$$

Combining equations (A.20)-(A.24) leads to the following PDE satisfied by the option price:

$$
\begin{aligned}
-f_t \;=\; & -rf + \left(r - Z^2 \mathrm{E}_Q \left[\frac{J_W^*}{J_W} Q \right] \right) S f_S \\
& + \left(\mu_Y + \kappa_Y Y + W \frac{J_{WW}}{J_W} \rho_{SY} \sigma_Y Y + \frac{J_{WY}}{J_W} \sigma_Y^2 + \frac{J_{WZ}}{J_W} \rho_{YZ} \sigma_Y \sigma_Z \right) f_Y \\
& + \left(\mu_Z + \kappa_Z Z + W \frac{J_{WW}}{J_W} \rho_{SZ} \sigma_Z Y + \frac{J_{WY}}{J_W} \rho_{YZ} \sigma_Y \sigma_Z + \frac{J_{WZ}}{J_W} \sigma_Z^2 \right) f_Z \\
& + \frac{1}{2} Y^2 S^2 f_{SS} + \frac{1}{2} \sigma_Y^2 f_{YY} + \frac{1}{2} \sigma_Z^2 f_{ZZ} + \rho_{SY} \sigma_Y Y S f_{SY} + \rho_{SZ} \sigma_Z Y S f_{SZ} \\
& + \rho_{YZ} \sigma_Y \sigma_Z f_{YZ} + Z^2 f \mathrm{E}_Q \left[\frac{J_W^*}{J_W} Q_f \right], \qquad \text{(A.25)}
\end{aligned}
$$

where we define $J_W^* \equiv J_W(W(1+Q), Y, Z, t)$. In the case of power utility function, the value function J has an analytical solution given by (A.2) and (A.4). Substitute this solution into (A.25) to get:

$$
\begin{aligned}
-f_t = & -rf + \left(r - Z^2 \mathrm{E}_Q \left[(1+Q)^{-\gamma} Q \right] \right) S f_S + \big[\mu_Y + \kappa_Y Y - \gamma \rho_{SY} \sigma_Y Y \\
& + \sigma_Y^2 \left(B_Y + 2C_{YY} Y + 2C_{YZ} Z \right) + \rho_{YZ} \sigma_Y \sigma_Z \left(B_Z + 2C_{ZZ} Z + 2C_{YZ} Y \right) \big] f_Y \\
& + \big[\mu_Z + \kappa_Z Z - \gamma \rho_{SZ} \sigma_Z Y + \rho_{YZ} \sigma_Y \sigma_Z \left(B_Y + 2C_{YY} Y + 2C_{YZ} Z \right) \\
& + \sigma_Z^2 Z \left(B_Z + 2C_{ZZ} Z + 2C_{YZ} Y \right) \big] f_Z + \frac{1}{2} Y^2 S^2 f_{SS} + \frac{1}{2} \sigma_Y^2 f_{YY} + \frac{1}{2} \sigma_Z^2 f_{ZZ} \\
& + \rho_{SY} \sigma_Y Y S f_{SY} + \rho_{SZ} \sigma_Z Y S f_{SZ} + \rho_{YZ} \sigma_Y \sigma_Z f_{YZ} + Z^2 f \mathrm{E}_Q \left[(1+Q)^{-\gamma} Q_f \right]. \qquad \text{(A.26)}
\end{aligned}
$$

On the other hand, under the risk-adjusted probability measure, $f(S, Y, Z, t)$ satisfies equation (A.6). Then relations (22)-(27) are verified by substituting them into (A.6) to get (A.26).

Monte Carlo Simulations

To verify the precision of the approximation of the conditional likelihood function $f_X(.)$ by the truncated Poisson-Normal mixture distribution, we conduct the following simulation exercise. Fixing the time horizon, Δt, to be 1 week as in our sample, we compare the conditional density functions of continuously-compounded weekly stock returns using the model (1)-(3) and the approximation (32)-(34). The true density function of model (1)-(3) is constructed by Monte-Carlo simulations. To be specific, we set the time increment in simulations to be 1/10 days and use a Euler approximation of the continuous time model to simulate 100,000 sample paths of S, Y, and Z. In simulations, the model parameters are those estimated by the QML method in the next section and the starting values of Y and Z are the sample averages of the implied time series of state variables.

The empirical density function of the simulated stock returns is plotted in Figure A.1 together with the density function of the truncated Poisson-Normal mixture distribution. It is obvious from the plots that the two density functions are very close. We conclude that the model (32)-(34) provides a good approximation of the jump-diffusion model.

We next examine if the QML estimation of the approximation model biases the parameters. We simulate samples of S, Y, and Z at weekly frequency for 366 weeks (matching the length of our sample). The time increment in simulations is again 1/10 days and we simulate 10,000 samples.For each sample, we perform the QML estimation of the model using the approximation (32)-(34). Table A.1 reports the sample statistics of the estimated parameters. Overall, the QML estimates are close to the true parameters.

Table A.1: **Monte-Carlo Analysis**

This table reports the statistics of the quasi maximum likelihood (QML) parameter estimates for simulated samples. We use the parameters reported in Table 2 (shown in the first column for comparison purpose) as the true model parameters to simulate sample paths of S, Y, and Z of length 366 weeks. We fix the time increment in simulations to be 1/10 days. For each simulated sample, we perform the QML estimation to the model of truncated Poisson-Normal mixture distribution of (32)-(34). We repeat this 10,000 times. We report the mean and standard deviation of parameter estimates.

	θ	Mean	Std.
μ_Y	2.841	2.569	0.333
κ_Y	-18.079	-16.356	2.002
σ_Y	0.334	0.290	0.011
μ_Z	7.745	7.782	1.436
κ_Z	-9.436	-9.474	1.596
σ_Z	1.529	1.427	0.053
μ_Q	-0.098	-0.108	0.077
σ_Q	0.160	0.136	0.109
ρ_{SY}	-0.495	-0.440	0.170
ρ_{SZ}	-0.597	-0.527	0.193
ρ_{YZ}	0.168	0.167	0.051

Figure A.1: **Monte-Carlo Simulations**

This figure shows the plots of the density functions of the approximate truncated Poisson-Normal mixture distribution and the empirical distribution of Monte-Carlo simulations for continuously compounded weekly stock returns. The model parameters are those reported in Table 2, and the starting values for Y and Z are those of the sample averages reported in Table 3. For the simulations, the time step is chosen to be 1/10 days, and the simulation horizon is five trading days. We repeat this 100,000 times.