Professor Zipf goes to Wall Street*

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Abstract

The heavy-tailed distribution of firm sizes first discovered by Zipf (1949) is one of the best established empirical facts in economics. We show that it has strong implications for asset pricing. Due to the concentration of the market portfolio when the distribution of the capitalization of firms is sufficiently heavy-tailed, an additional risk factor generically appears even for very large economies. Our two-factor model is as successful empirically as the three-factor Fama-French model.

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1 Introduction

Zipf (1949) discovered that when sizes of US corporation assets are ranked from the largest to the smallest, the firm size $s(n)$ of the n^{th} largest firm is inversely proportional to its rank n, i.e., $s(n) \sim 1/n$. This distribution is now referred to as the Zipf's law.[1] Zipf's law is robust (Ijri and Simon 1977) and has been confirmed in different countries (Ramsden and Kiss-Haypa 2000) and with several measures of firm size including number of employees, profits, sales, value added, and market capitalizations (Axtell 2001, Axtell 2006, Gabaix *et al.* 2006, Marsili 2005, Simon and Bonini 1958).

The Zipf distribution of firm sizes implies that the market portfolio is poorly diversified in the sense that a few companies account for a very large part of the overall market capitalization. For instance, the top ten largest companies represent between one fifth and one fourth of the entire US market capitalization. This concentration of the market has strong implications for the Arbitrage Pricing Theory of Ross (1976).

Consider a factor model where one factor (e.g., the market) is the return of a portfolio of assets. By assumption, the return of that portfolio has only factor risk and no residual risk. This implies a linear constraint on the residuals of all assets, namely that their sum weighted according to the factor-portfolio is equal to zero. This creates correlation in the residuals, as already recognized by Fama (1973) and Sharpe (1990, footnote 13) when the return on the market portfolio is considered the only explaining factor, or by Chamberlain (1983) in the case where there exist several linearly independent portfolios that contain only factor risk. The residual correlations are equivalent to the existence of at least one factor in the residuals that is uncorrelated with the market and the other factors. The impact of this new factor is usually neglected on the basis of the law of large numbers applied to well-diversified portfolios.

However, when the distribution of the weights of the portfolios replicating the factors — the distribution of the capitalization of firms in the case of the market portfolio — is sufficiently heavy-tailed, the law of large numbers that implies the diminishing contribution of the residual risk in the total risk of "well-diversified portfolios" (Ross 1976, Huberman 1982) breaks down. Intuitively, the largest firms contribute idiosyncratic risks that cannot be diversified away even when the number of firms is very large. In this case, the generalized

[1] Inverting this relation, we have that the rank of the n^{th} largest firm is inversely proportional to its size $n \sim 1/s(n)$, which is the complementary cumulative Pareto distribution with a tail exponent $\mu = 1$.

central limit theorem (Gnedenko and Kolmogorov 1954) shows that the impact of the factor in the residuals does not vanish even for infinite economies.[2] We term the factor in the residuals the Zipf factor and show that it is responsible for a significant amount of risk for portfolios that would have been otherwise assumed "well-diversified" in its absence. Correspondingly, the Zipf factor adds an extra risk premium to the asset pricing relation.

We show that a simple proxy for the Zipf factor is the difference in returns between the equal-weighted and the value-weighted market portfolios. We test the Zipf model with size and book-to-market double-sorted portfolios as well as industry portfolios. We find that the Zipf model performs as well as the Fama-French model in terms of the magnitude and significance of pricing errors and explanatory power, despite that it has only two factors instead of three.

2 Theory

2.1 The distribution of firm size

The Zipf law is a special case of the Pareto distribution. Given an economy of N firms, whose sizes S_i, $i = 1, \ldots, N$, follow a Pareto law with tail index μ, the ratio of the capitalization of the largest firm to the total market capitalization

$$w_N = \frac{\max S_i}{\sum_{i=1}^{N} S_i} \ , \tag{1}$$

which is nothing but the weight of the largest company in the market portfolio, behaves on average like

$$\mathrm{E}\left[w_N\right] \longrightarrow 0, \quad \text{if } \mu > 1, \tag{2}$$

$$\mathrm{E}\left[1/w_N\right] \longrightarrow \frac{1}{1-\mu}, \quad \text{if } \mu < 1, \tag{3}$$

as the number of firms N goes to infinity (Bingham *et al.* 1987).

[2] In a different context, Gabaix (2005) has proposed that the same kind of argument can explain that idiosyncratic firm-level fluctuations are responsible for an important part of aggregate shocks, and therefore provide a microfoundation for aggregate productivity shocks. Indeed, as in the present article, it is suggested that the traditional argument according to which individual firm shocks average out in aggregate breaks down if the distribution of firm sizes is fat-tailed.

This result means that when the distribution of firm sizes admits a finite mean (i.e. $\mu > 1$), the weight of the largest firm in the market portfolio goes to zero, and so do the weights of any other firms, in the limit of a large market. In terms of asset pricing, as we show below, the fact that the weight of each individual firm in the economy is infinitesimal ensures that the APT equation holds for each asset and not only on average (Connor 1982). In contrast, when the distribution of firm sizes has no finite mean (i.e. $\mu \leq 1$), equation (3) shows that the asymptotic weight of the largest firm in the market portfolio does not vanish and, for such an economy, the market portfolio is not well diversified. A practical consequence is that the APT equation, if it holds, can only hold on average, with possibly large pricing errors for individual assets.

In order to get a closer look at the concentration of the market portfolio, we focus on its Herfindahl index, which is perhaps the most widely used measure of economic concentration (Polakoff 1981, Lovett 1988),

$$H_N = ||w_m||^2 = \sum_{i=1}^{N} w_{m,i}^2 \ , \tag{4}$$

where $w_{m,i}$ denotes the weight of asset i in the market portfolio whose composition is given by the N-dimensional vector w_m, with $\sum_{i=1}^{N} w_{m,i} = 1$. The Herfindahl index takes into account the relative size and distribution of the firms traded in the market. It approaches zero when the market consists of a large number of firms with comparable sizes. It increases both as the number of firms in the market decreases and as the disparity in size between those firms increases. Our use of the Herfindahl index is not only guided by common practice but also by its superior ability to provide meaningful information about the degree of diversification of an unevenly distributed stock portfolio (Woerheide and Persson 1993).[3] We say that a portfolio is *well-diversified* if its Herfindahl index goes to zero when the number N of firms traded in the market goes to infinity.

For illustration purpose, let us first concentrate on an economy where the sizes, sorted

[3]Even if its relevance has been sometimes questioned, in particular when the distribution of weights has an infinite second moment (Mandelbrot 1997), as it is the case when the distribution of firm sizes follows Zipf's law, we will see below that this choice of concentration measure of a portfolio is not arbitrary but is instead dictated by the choice of the risk measure taken as the variance of the portfolio returns.

in descending order, of the N firms are deterministically given by

$$S_{i,N} = \left(\frac{i}{N}\right)^{-1/\mu} . \tag{5}$$

We have arbitrarily chosen the size of the smallest firm to be equal to one. Alternatively, we can think of $S_{i,N}$ as the size of the i^{th} largest firm relative to the size of the smallest one. With this simple model, the rank i of the i^{th} largest company is directly proportional to its size taken to the power of minus μ and the distribution of sizes obeys a Pareto law with a tail index of μ. It can be shown that the weight of the largest firm in the market portfolio goes to zero, as N goes to infinity, when μ is larger than or equal to one while it goes to some positive constant when μ is less than one. More precisely, we have

$$w_{m,1} \longrightarrow 0, \quad \text{if } \mu \geq 1, \tag{6}$$

$$w_{m,1} \longrightarrow \frac{1}{\zeta\left(1/\mu\right)}, \quad \text{if } \mu < 1, \tag{7}$$

where $\zeta(1/\mu) = \sum_{n=1}^{\infty} n^{-1/\mu}$ denotes the Riemann zeta function.

For the Herfindahl index, one gets

$$H_N = \begin{cases} \dfrac{1}{1-\frac{1}{(1-\mu)^2}} \cdot \dfrac{1}{N} + O\left(N^{2/\mu-2}\right), & \mu > 2, \\ \dfrac{\ln N + \gamma}{4N} + O\left(N^{-3/2} \ln N\right), & \mu = 2, \\ \left(\dfrac{1-\mu}{\mu}\right)^2 \zeta(2/\mu) \cdot \dfrac{1}{N^{2-2/\mu}} + O\left(N^{3(1/\mu-1)}\right), & 1 < \mu < 2, \\ \dfrac{\pi^2}{6} \dfrac{1}{(\gamma + \ln N)^2} + O\left(N^{-1}(\gamma + \ln N)^{-2}\right), & \mu = 1, \\ \dfrac{\zeta(2/\mu)}{\zeta(1/\mu)^2} + O\left(N^{1-1/\mu}\right), & \mu < 1. \end{cases} \tag{8}$$

In accordance with the behavior of the weight of the largest firm, H_N goes to zero when the index μ is larger than or equal to one, while it goes to some positive constant otherwise. However, the decay rate of H_N toward zero becomes slower and slower as μ approaches 1 (from above). In practice, when the number of traded firms is large but finite, the concentration of the market portfolio can remain significant even if μ is larger than one (specifically when μ lies between one and two).

To illustrate this distribution, the upper panel of Figure 1 depicts the value of the weight of the largest firm in the market portfolio while the lower panel shows the inverse of the Herfindahl index as a function of μ. The inverse of the Herfindahl index can be understood as the effective number of assets in the portfolio. It is the exact number of assets required to construct an *equally-weighted* portfolio with the same concentration (same Herfindahl index) as the original portfolio, since the Herfindahl index of any equally-weighted portfolio made of N assets is just $H = 1/N$. This allows us to interpret the inverse of the Herfindahl index as the effective number of assets of a portfolio. The solid curves show the limit situation of an infinite economy while the dotted and dash-dotted curves account for the finiteness of the economy. The dotted curve refers to the case where only one thousand companies are traded while the dash-dotted curve corresponds to an economy with ten thousand firms. The lower panel shows that the number of effective assets, in a market where about one thousand to ten thousand assets are traded, ranges between 35 and 60 if the distribution of market capitalizations follows Zipf's Law ($\mu = 1$). This observation remains robust to slight departure from $\mu = 1$. Clearly, there is a substantial difference between an economy with a large number of assets trades as is the case of the US economy and the case where there is an infinite number of assets.

To be a little more general, we can consider an economy where the firm sizes are randomly drawn from a power law distribution of size. Proposition 2 in Appendix A focuses on this situation in detail and shows no qualitative changes with respect to the result (8) derived for deterministic firm sizes. Concretely, for an economy in which the distribution of firm sizes follows Zipf's law, we obtain a typical value of H_N of about 5% for a market where $8,000$ assets are traded.[4] This value is much larger than the concentration index of the equally-weighted portfolio of all assets which would be of the order of 0.012%. Intuitively, $H_N \simeq 5\%$ means that there are only about $1/H_n \simeq 20$ effective assets in a supposedly well-diversified portfolio of $8,000$ assets. This order of magnitude is the same as the one obtained above where the distribution of firm sizes was assumed to follow a deterministic sequence.

[4] These figures are compatible with the number of stocks currently listed on the Amex, the Nasdaq and the NYSE.

2.2 Correlated residuals in factor models

Consider an economy with N firms with stock returns determined according to the following factor model

$$r = \alpha + \beta_m \cdot (r_m - \mathrm{E}[r_m]) + \varepsilon, \tag{9}$$

where

- r is the random $N \times 1$ vector of asset returns;
- $\alpha = \mathrm{E}[r]$ is the $N \times 1$ vector of asset return mean values. We do not make any assumption on the *ex-ante* mean-variance efficiency of the market portfolio or on the absence of arbitrage opportunity, so that α is not, *a priori*, specified;
- r_m is the random return on the market portfolio;
- β_m is the $N \times 1$ vector of the stocks' loadings on the market factor;
- ε is the random $N \times 1$ vector of disturbance terms with zero average $\mathrm{E}[\varepsilon] = 0$ and covariance matrix $\Omega = \mathrm{E}[\varepsilon \cdot \varepsilon']$, where the prime denotes the transpose operator. The disturbance terms are assumed to be uncorrelated with the market return r_m and the factors ϕ_i.

We posit a single factor equal to the market portfolio return for simplicity; all our results hold if there were additional risk factors in the model.

It would be natural to assume that (i) Ω is diagonal in order to interpret the ε as the specific risk of the assets but, as we shall see below, there is an internal consistency condition that makes this impossible and forces the disturbances ε to be correlated. A weaker hypothesis on Ω would be that (ii) all its eigenvalues are uniformly bounded from above by some constant λ independent of the size of the economy. This implies that the covariance matrix of the stock returns defined as

$$\Sigma = \mathrm{E}\left[(r-\alpha)(r-\alpha)'\right] = \beta\beta' \cdot \mathrm{Var}[r_m] + \Omega, \tag{10}$$

has an approximate factor structure, according to the definition in Chamberlain (1983) and Chamberlain and Rothschild (1983). But these two assumptions (i) and (ii) are in fact equivalent, as shown by Grinblatt and Titman (1985). Indeed, a simple repackaging of the

N security returns into N new returns constructed by forming N portfolios of the primitive assets allows us to get a new formulation of expression (9) with mutually uncorrelated disturbance terms.

To understand why the disturbance terms cannot be uncorrelated, let us first denote by w_m the vector of the weights of the market portfolio. Accounting for the fact that the market factor is itself composed of the assets that it is supposed to explain, the model must necessarily fulfill the internal consistency relation

$$r_m = w'_m \cdot r. \tag{11}$$

Left-multiplying (9) by w'_m, the internal consistency condition (11) implies the following relation

$$({w'}_m \cdot \beta - 1) \cdot (r_m - \mathrm{E}[r_m]) + {w'}_m \cdot \varepsilon = 0 \ . \tag{12}$$

Then, by our assumption of absence of correlation between r_m and ε, it follows trivially that

$$w'_m \cdot \varepsilon = 0 \qquad \text{almost surely,} \tag{13}$$

while

$$w'_m \cdot \beta = 1. \tag{14}$$

An important consequence of this result is the breakdown of the standard assumption of independence (or, at least, of the absence of correlation) between the non-systematic components of the returns of securities, pointed out by several authors (Fama 1973, Sharpe 1990). This correlation between the disturbance terms may *a priori* pose problems in the pricing of portfolio risks: the systematic risk of a portfolio is totally captured by its exposure to the market portfolio if (and only if) the disturbance terms can be averaged out by diversification. Previous authors have suggested that this is indeed what happens in economies in the limit of a large market $N \to \infty$, for which the correlations between the disturbance terms are expected to vanish asymptotically and the internal consistency condition seems irrelevant. For example, while Sharpe (1990, footnote 13) concluded that, as a consequence of equation (13), at least two of the disturbances, say ε_i and ε_j, must be negatively correlated, he suggested that this problem would disappear in economies with infinitely many securities. Actually, contrary to this belief, we show below that even for economies with infinitely many securities, when the companies exhibit a fat-tailed distribution of sizes as they do in reality, the constraint (13) leads to the important consequence that the risk a well-diversified portfolio does not reduce to its market risk even in the limit of a very large economy. A significant proportion of asset-specific risk remains which cannot be diversified away by the simple aggregation of a very large number of assets.

The fact that the disturbance terms ε in the market model (9) are correlated according to the condition (13) means that there exists at least one common factor z in the residuals, so that ε can be expressed as

$$\varepsilon = \gamma \cdot z + \eta \ , \tag{15}$$

where γ is the vector of loading of the factor z.[5] For simplicity, we choose η to be a vector of *uncorrelated* residuals with zero mean.[6] Since $w_m' \varepsilon = 0$, z and η are not independent from one another and we have

$$z = -\frac{w_m' \eta}{w_m' \gamma}, \tag{16}$$

provided that $w_m' \gamma \neq 0$. Therefore, in this framework, z is not actually a factor in the usual sense of the term since it is correlated with η. We refer to it as an "endogenous" factor. The market model (9) then becomes

$$r = \alpha + \beta \cdot (r_m - \mathrm{E}[r_m]) + \gamma \cdot z + \eta, \tag{17}$$

with

- $\mathrm{Cov}(r_m, z) = \mathrm{Cov}(r_m, \eta) = 0$, from the absence of correlation between r_m and ε;
- $\mathrm{Var}[\eta] = \Delta$, where Δ is a diagonal matrix;
- $\mathrm{Var}[z] = \frac{w_m' \Delta w_m}{(w_m' \gamma)^2}$;
- $\mathrm{Cov}(z, \eta) = -\frac{1}{w_m' \gamma} \cdot w_m' \Delta$.

[5] Our only requirement is that the covariance matrix of ε exhibits an eigenvalue that goes to infinity in the limit of an infinite economy, when H_N does not go to zero. In contrast, when H_N goes to zero as $N \to \infty$, the largest eigenvalue should remain bounded. This requirement derives simply from the results of Chamberlain (1983) and Chamberlain and Rothschild (1983), who have linked the existence of K unbounded eigenvalues (in the limit $N \to \infty$) of the covariance matrix of the asset returns to a unique approximate factor structure, such that the K associated eigenvectors converge and play the role of K factor loadings.

[6] It would be enough to assume that all the eigenvalues of the covariance matrix of η are positive and uniformly bounded by some positive constant (Grinblatt and Titman 1983).

Below we show that factor z matters for asset pricing when the distribution of firm sizes is fat tailed, even when the number of firms goes to infinity. We therefore name factor z the *Zipf factor*.

To show that the correlation between two disturbance terms ε_i and ε_j is not negligible in an infinite size market, we can evaluate their typical magnitude. To simplify the notation, and without loss of generality, rescale the vector γ by $w_m'\gamma$, so that the relation (16) becomes

$$z = -w_m'\eta, \tag{18}$$

with $w_m'\gamma = 1$. The covariance matrix Ω of ε is

$$\Omega = (w_m'\Delta w_m)\,\gamma\gamma' - \gamma w_m'\Delta - \Delta w_m\gamma' + \Delta. \tag{19}$$

Assuming, for instance, that all the γ_i's are equal to one (the condition $w_m'\gamma = 1$ is then automatically satisfied from the normalization of the weights w_m), the correlation between ε_i and ε_j $(i \neq j)$ reads

$$\rho_{ij} = \frac{H_N - w_{m,i} - w_{m,j}}{\sqrt{(1 + H_N - 2w_{m,i})\,(1 + H_N - 2w_{m,j})}}, \tag{20}$$

$$= \frac{H_N}{1 + H_N} \cdot \left(1 + O(w_{m,i(j)}/H_N)\right) \; . \tag{21}$$

Expression (21) shows that, provided the market portfolio is sufficiently well-diversified so that the weight of each asset and the concentration index goes to zero in the limit of a large market ($N \to \infty$), the correlations ρ_{ij} between any two disturbance terms go to zero as usually assumed. However, as soon as H_N goes to zero more slowly than $1/N$, the largest eigenvalue of the correlation matrix, associated with the (asymptotic) eigenvector $1 = (1, 1, \ldots, 1)'$, is $\lambda_{\max,N} \simeq N \cdot \frac{H_N}{1+H_N}$ and goes to infinity as the size of the economy grows indefinitely. This clearly shows that the correlations between the disturbance terms will generally be important when the distribution of firm sizes has fat tails.

The question that we now have to address is whether these correlations challenge the usual assumption that well-diversified portfolios have only factor risk and no residual risk. For this, let us consider a well diversified portfolio w_p, *i.e.*, a portfolio such that $||w_p||^2 \to 0$ as the size of the economy goes to infinity. From equation (19), the residual variance of this portfolio, namely the part of the variance of the portfolio that is not ascribed to systematic risk factors, reads

$$w_p' \Omega w_p = (w_m' \Delta w_m) \left(\gamma w_p'\right)^2 - 2\,(w_m' \Delta w_p)\,(\gamma' w_p) + w_p' \Delta w_p' \ . \tag{22}$$

In addition to our previous hypothesis that Δ is a diagonal matrix, we assume that its entries are uniformly bounded from below by some positive constant c_1 and from above by some constant $c_2 < \infty$ and that $|\gamma w_p'|$ is uniformly bounded from below by some positive constant d_1 and from above by some finite constant d_2 (this is the case, for instance, when one considers γ equal to a vector of ones). Then

$$w_p' \Delta w_p' \ \leq \ c_2 \cdot ||w_p||^2 \to 0, \tag{23}$$

$$|(w_m' \Delta w_p)\,(\gamma' w_p)| \ \leq \ c_2 \cdot d_2 \cdot ||w_m|| \cdot ||w_p|| \to 0, \tag{24}$$

and

$$c_1 \cdot d_1 \cdot ||w_m||^2 \leq (w_m' \Delta w_m) \left(\gamma w_p'\right)^2 \leq c_2 \cdot d_2 \cdot ||w_m||^2, \tag{25}$$

so that

$$w_p' \Omega w_p \sim K \cdot H_N, \quad K > 0, \quad \text{as} \quad N \to \infty. \tag{26}$$

Therefore, the residual variance $w_p' \Omega w_p$ of any "well-diversified portfolio" w_p goes to zero, as the size N of the economy goes to infinity, *if and only if* the concentration index H_N of the market portfolio goes to zero. In the case of a real economy with $\mu = 1$, Proposition 2 of Appendix A shows that the Herfindahl index H_N of the market portfolio goes to zero but at the particularly slow decay rate of $1/(\ln N)^2$. As a consequence, the residual variance may still account for a significant part of the total portfolio variance.

2.3 The Zipf factor and asset pricing

In his article establishing the arbitrage pricing theory, Ross (1976, p. 347) explicitly assumes that the disturbance terms in the factor model (9) are "mutually stochastically uncorrelated," which is inconsistent with the constraint (13) if we assume that the factors (or at least some of them) can be replicated by portfolios. Indeed, the derivation of the APT results from the construction of a *well-diversified* arbitrage portfolio (step 1 in Ross (1976, p. 342)) chosen to have no systematic risk (step 2). The fact that this arbitrage portfolio has no specific risk in the limit of a large number of assets (law of large numbers) conditions the results of steps 3 and 4. Unfortunately, as shown in the previous section, if one of the factors (e.g., the market) is replicated by a portfolio whose weights are distributed according to a sufficiently fat-tailed distribution, the specific risk of this portfolio cannot be diversified away. In that case, the conclusion from steps 3 and 4 in Ross (1976) breaks down.

Alternatively, some authors have derived pricing results when the residuals exhibit correlation. In particular, Chamberlain (1983) and Chamberlain and Rothschild (1983) have developed the appropriate formalism to deal with this problem, while Stambaugh (1982) and Ingersoll (1984) have provided sharp pricing bounds in the presence of correlation between the error terms. Basically, when all the eigenvalues of the residual covariance matrix are bounded as more and more assets are added to the market, the APT still holds. In contrast, when some eigenvalues grow without bound, the factors associated with these eigenvalues must be split off from the residuals and considered as new explaining factors that potentially are priced. This argument is at the basis of the choice of the specification (15) of the dependence structure of the disturbances of our market model. Therefore, if we explicitly include our Zipf factor z in the analysis, the original derivation of Ross' results still holds, as shown by Chamberlain (1983). Indeed, a key technical assumption for the APT to hold is that the ε's (in equation (9)) are "sufficiently independent to ensure that the law of large numbers holds" (Ross 1976, p. 342) and, as explained in the previous sections, this condition breaks down. Nonetheless, this condition holds for the residuals η defined by equations (15-17). Then, for the one factor model (17), we obtain the following result.

Proposition 1 *Consider a market where N assets are traded and for which the internal consistency condition (13) holds, so that the returns of the set of assets obey the following dynamics: $r = \mathrm{E}[r] + \beta \cdot (r_m - \mathrm{E}[r_m]) + \gamma \cdot z + \eta$, where z is the (zero-mean) additional factor resulting from the internal consistency condition and r_m is uncorrelated with z and with the centered disturbance vector η. Then, under the usual assumptions required for the APT to hold, the expected return on asset i satisfies*

$$\mathrm{E}[r_i] - r_f = \beta_i \cdot (\mathrm{E}[r_m] - r_f) + (\gamma_i - \gamma_m \cdot \beta_i) \cdot (\mathrm{E}[r_z] - r_f) \ , \tag{27}$$

where r_f denotes the risk free interest rate and $\mathrm{E}[r_z] \geq r_f$ is the expected return on any portfolio w_z with no market exposure, $w_z' \cdot \beta = 0$, with unit exposure to the Zipf factor z, $w_z' \cdot \gamma = 1$, and which is well-diversified in the sense that the variance of the new residuals $\mathrm{Var}[w_z \cdot \eta]$ goes to zero as the number N of assets goes to infinity. $\gamma_m = w_m' \cdot \gamma$ is the gamma *of the market portfolio.*

The proof of this result proceeds as follows. Starting from the model (17) and following step by step the demonstration of theorems I and II in Ross (1976), we get the asymptotic result

$$\mathrm{E}\left[r\right] = \rho\iota + \lambda_1 \beta + \lambda_2 \gamma, \tag{28}$$

where ρ, λ_1 and λ_2 are three non-negative constants and ι denotes a vector of ones. Their values are determined by the expected return of the market portfolio w_m, of the portfolio w_z and of any other well-diversified portfolio with no systematic risk. This leads to identifying ρ with r_f, λ_2 with $(\mathrm{E}\left[r_z\right] - r_f)$ and λ_1 with $-\gamma_m \cdot (\mathrm{E}\left[r_z\right] - r_f) - r_f$. The quantity $\gamma_m = w_m' \cdot \gamma$ never vanishes, due to the dependence of the residuals.

Two comments are in order. First, expression (27) looks like a standard APT decomposition of the risk premia of the expected return of a given asset i weighted by their factor loading, except for one important feature: the risk premium due to the Zipf factor has its amplitude controlled by the factor loading γ_i (as usual) *corrected* by the unusual term $-\gamma_m \beta_i$. In a standard factor decomposition, it is always convenient to impose $\gamma_m \equiv \vec{w}_m' . \vec{\gamma} = 0$ so that the contribution to the total risk premium due to any factor is proportional to its corresponding factor loading γ_i. In the case of the Zipf factor, this is intrinsically impossible. In this sense, expression (27) is not the result of a standard factor decomposition. The pricing equation (27) highlights the contribution of the Zipf factor to the total risk premium of a given asset. As we shall see below, the fact that the factor loading β_i on the market portfolio contributes to the amplitude of the risk premium due to the Zipf factor provides an interesting interpretation of the book-to-market effect.

Second, when the market portfolio is well-diversified, the contribution of the Zipf factor vanishes asymptotically so that the risk premium associated with this risk factor goes to zero in the limit of an infinitely large market.

The pricing formula given by proposition 1 offers an interesting new insight into the valuation of asset prices. However, to take it to the data, we still need to identify empirically the Zipf factor. There are many ways to do this. We could, for instance, identify the Zipf factor from a factor analysis of the residuals of the market model. However, we follow a simpler route. Recall that the risk premium associated with the Zipf factor is due to the exposure of well-diversified portfolios to residual risk. Therefore, well-diversified portfolios such as the equally-weighted portfolio are particularly sensitive to this risk. We therefore use as a simple proxy for the Zipf factor the difference in returns between the equal-weighted and the value-weighted market portfolios. The numerical simulations presented in section B of the Appendix show that this choice does a good job of capturing the systematic risk in the residuals of the market model and the corresponding risk premium. We also checked (results not reported) that this proxy for the Zipf factor correlates highly with the first principal component of the residuals from the market model.

3 Empirics

This section examines the empirical performance of the Zipf model, contrasting it with the market model and the Fama-French model. Table 1 offers summary statistics for the value-weighted and equal-weighted market portfolios, the Zipf factor, and the Fama-French size (SMB) and value (HML) factors (see Fama and French (1993) for details on the construction of these two portfolios). The Sharpe ratios of the Zipf factor is 0.11 (0.39 annualized), of the same magnitude as the market or the HML portfolios and higher than the SMB portfolio. The Zipf factor's correlation with the market is 0.38, which is of the same order as the correlations between the factors SMB and HML and the market factor (0.33 and 0.22 respectively). The Zipf portfolio is highly correlated with the SMB portfolio as would be expected.

We test the Zipf model with the following time-series regression:

$$r_{i,t} - r_{f,t} = \alpha_i + \beta_i \cdot (r_{m,t} - r_{f,t}) + \gamma_i \cdot (r_{z,t} - r_{f,t}) + \varepsilon_i(t) \ . \tag{29}$$

We also estimate similar time-series regressions for the market and the Fama-French models. As test assets, we use the monthly excess returns of twenty-five value-weighted portfolios sorted by the quintiles of the distribution of size and book-to-market and the returns of thirty value-weighted industry portfolios.[7] Tables 2 and 3 present our results for the period from July 1931 to December 2005.

Each panel of Table 2 shows alphas, corresponding t-statistics, and R^2 for the test

[7] We have used the monthly data available on Professor French's website for the 25 portfolios sorted by size and book-to-market, the thirty industry portfolios, the market factor, the risk-free interest rate, and the factors SMB and HML.

portfolios under each asset pricing model. The number below each square array of numbers is the average of the absolute values above. The mean absolute alpha is 0.22 for the market model, 0.16 for the Zipf model, and 0.13 for the Fama-French model. The average mispricing is thus considerably lower in the Zipf and Fama-French models than in the market model. There is very little difference between the Zipf and Fama-French models. This is remarkable since the Zipf model only has two factors whereas Fama-French has three and because of the remark in Lewellen *et al.* (2006) that the Fama-French factors are almost guaranteed to perform well for the set of the 25 double sorted portfolios because they have a strong factor structure (the three Fama-French factors explain more than 90% of the variation of the time-series of the portfolios' returns). The cross-section structure of the pricing errors is similar in both models, with the largest mispricings occurring for small value firms. Indeed, these are the portfolios where the alphas are statistically significant as can be assessed from the t-statistics. The average R^2 across the test portfolios is 0.77 in the market model, 0.86 in the Zipf model, and 0.91 in the Fama-French model. We find that the Zipf factor adds substantial explanatory power for the time series of returns of the test portfolios relative to the market model.

Table 3 presents similar statistics for industry portfolios. Here we see that the performance of the three models is substantially the same. The mean absolute alphas are 0.15, 0.17, and 0.19 for the market model, the Zipf model, and the Fama-French model, respectively. And, as can be seen from the t-statistics, the three models fail in almost the same industries. The explanatory power of the three models is also similar to each other, with R^2 ranging from 0.64 to 0.67, much lower than for the size and book-to-market portfolios.

4 Conclusion

Starting from a model in which the only a priori systematic risk is the market portfolio, we show that there is a new source of significant systematic risk that should be priced. This new risk factor arises from a simple internal consistency condition whereby the market portfolio is made of the very assets whose returns it is supposed to explain. The new factor becomes important when the distribution of the capitalization of firms is sufficiently fat-tailed as is the case of real economies as documented abudantly since Zipf (1949). We therefore term the new factor the Zipf factor. We show that our two-factor Zipf model performs empirically as well as the three-factor Fama-French model in the cross-section of stocks.

Appendix

A Concentration of the market portfolio when the distribution of firm sizes follows a power law

We consider an economy where firm sizes are randomly drawn from a power law distribution. By application of the generalized law of large numbers (Feller 1971, Gnedenko and Kolmogorov 1954, Ibragimov and Linnik 1975) and using standard results on the limit distribution of self-normalized sums (Darling 1952, Logan *et al.* 1973), we can state the following result.[8]

Proposition 2 *The asymptotic behavior of the concentration index* H_N *is the following:*

1. *provided that* $\mathrm{E}[S^2] < \infty$*,*

$$H_N = \frac{1}{N} \frac{\mathrm{E}\left[S^2\right]}{\mathrm{E}\left[S\right]^2} + o_p(1/N);$$

2. *provided that* S *is regularly varying with tail index* $\mu = 2$ *and* $s^\mu \cdot \Pr\left[S > s\right] \to c$ *as* $s \to \infty$*,*

$$H_N = \frac{c}{\mathrm{E}\left[S\right]^2} \frac{\ln N}{N} + o_p\left(\frac{1}{N \ln N}\right);$$

3. *provided that* S *is regularly varying with tail index* $\mu \in (1,2)$ *and* $s^\mu \cdot \Pr\left[S > s\right] \to c$ *as* $s \to \infty$*,*

$$H_N = \left[\frac{\pi c}{2\Gamma\left(\frac{\mu}{2}\right) \sin \frac{\mu\pi}{4}}\right]^{2/\mu} \frac{1}{\mathrm{E}\left[S\right]^2} \cdot \frac{1}{N^{2-2/\mu}} \cdot \xi_N + o_p\left(\frac{1}{N^{2-2/\mu}}\right) ,$$

where ξ_N *is a positive free parameter characteristic of the distribution of firm sizes in the market under consideration. In the limit of large markets, the unconditional*

[8] For simplicity, we have assumed that the firm sizes S_i are independent. Proposition 2 can however be generalized to the more realistic case where firm sizes are not independent. Under mild mixing conditions, the results remain the same up to a scale factor (Jakubowski 1993, Davis and Hsing 1995).

distribution of this parameter is the stable law $S(\mu/2, 1)$*;*[9]

4. *provided that* S *is regularly varying with tail index* $\mu = 1$ *and* $s^{\mu} \cdot \Pr[S > s] \to c$ *as* $s \to \infty$*,*

$$H_N = \frac{\pi}{2 \cdot \ln^2 N} \cdot \xi_N + O_p\left(\frac{1}{\ln^3 N}\right),$$

where ξ_N *is a positive free parameter characteristic of the distribution of firm sizes in the market under consideration. In the limit of large markets, the unconditional distribution of this parameter is the Lévy law* $S(1/2, 1)$*;*

5. *provided that* S *is regularly varying with tail index* $\mu \in (0, 1)$ *and* $s^{\mu} \cdot \Pr[S > s] \to c$ *as* $s \to \infty$*,*

$$H_N = \frac{4}{\pi^{1/\mu}} \left[\Gamma\left(\frac{1+\mu}{2}\right) \cos\frac{\pi\mu}{4}\right]^{2/\mu} \cdot \xi_N,$$

where ξ_N *is a positive free parameter characteristic of the distribution of firm sizes in the market under consideration. In the limit of large markets, the unconditional distribution of this parameter is given by the limit law of the ratio* $\frac{\zeta_N}{\zeta'^2_N}$*, where* ζ_N *and* ζ'_N *denote two sequences of strongly correlated positive random variables that converge in law to* $S(\mu/2, 1)$ *and* $S(\mu, 1)$ *respectively;*[10]

6. *provided that* S *is slowly varying*[11]*,*

$$H_N \to 1, \quad a.s.$$

As a consequence of the fourth statement of the proposition above, for economies in which the distribution of firm sizes follows Zipf's law ($\mu = 1$) the asymptotic behavior of the concentration index H_N of the market portfolio is given by

$$H_N \simeq \frac{\pi}{2 \cdot (\ln N)^2} \cdot \xi_N, \tag{30}$$

[9] The stable law $S(\alpha, \beta)$ has characteristic function $\psi_{\alpha,\beta}(s) = \begin{cases} \exp\left[-|s|^{\alpha} + is\beta \tan\frac{\alpha\pi}{2}|s|^{\alpha-1}\right] & \alpha \neq 1, \\ \exp\left[-|s| - is\beta\frac{2}{\pi} \cdot \ln s\right] & \alpha = 1, \end{cases}$ with $\beta \in [-1, 1]$.

[10] More precisely, the sequence of random vectors $(\xi_N, \zeta_N)'$ converges to an *operator*-stable law with stable marginal laws $S(\mu/2, 1)$ and $S(\mu, 1)$ respectively, and a spectral measure concentrated on arcs $\pm(x, x^2)$. The full characterization of the spectral measure is beyond the scope of this article (see (Meerschaert and Scheffler 2001, Section 10.1) for details).

[11] The random variable S is slowly varying if its distribution function F satisfies $\lim_{x\to\infty} \frac{1-F(tx)}{1-F(x)} = 1$, for all $t > 0$. It corresponds to the limit case where S is regularly varying with $\mu \to 0$.

where ξ_N is a sequence of positive random variables with stable limit law $S(1/2, 1)$, namely the Lévy law with density

$$f(x) = \frac{1}{\sqrt{2\pi}} \cdot x^{-3/2} \mathrm{e}^{-\frac{1}{2x}}, \quad x \geq 0. \tag{31}$$

This shows that, even if the concentration of the market portfolio goes to zero in the limit of an infinite economy, it goes to zero extremely slowly as the size N of the economy diverges. Accounting for the fact that the numeric factor ξ_N in (30) is a specific realization (characteristic of the state of the market under consideration) of a random variable with asymptotic law given by the Lévy law (31) whose median value is approximately equal to 2.198, a typical value of H_N is $4-5\%$ for a market where $7,000$ to $8,000$ assets are traded. This value is much larger than the concentration index of the equally-weighted portfolio which would be of the order of $0.012 - 0.014\%$. Intuitively, $H_N \simeq 4-5\%$ means that there are only about $1/H_n \simeq 20-25$ effective assets in a typical portfolio supposedly well-diversified on $7,000$ to $8,000$ assets. This order of magnitude is the same as the one obtained in the example where the distribution of firm sizes was assumed to follow a deterministic sequence.

B Analysis of synthetic markets generated numerically

In order to assess the impact of the internal consistency factor in real stock markets of finite size, we present in table 4 the results of numerical simulations of synthetic markets with respectively $N = 1,000$ and $N = 10,000$ traded assets. We construct the synthetic markets according to the market model (17) where the only *explicit* risk factor is the market but taking into account the dependence in the residuals. We take the initial distribution of the capitalization of firms to be the Pareto distribution

$$\Pr\left[S \geq s\right] = \frac{1}{s^{\mu}} \cdot 1_{s \geq 1} \ . \tag{32}$$

We investigate various synthetic markets characterized by different tail indices μ, from $\mu = 1/2$ (deep in the heavy-tailed regime), $\mu = 1$ (borderline case often referred to as the Zipf law when expressed with sizes plotted as a function of ranks), to $\mu = 2$ (for which the central limit theorem holds and standard results are expected). It is important to stress that the results presented in table 4 are insensitive to the shape of the bulk of the distribution of firm sizes, and only the tail $\Pr[S \geq s] \sim s^{-\mu}$, for large s, matters.

The three values of the tail index μ equal to 2, 1 and 1/2 correspond to the three major behaviors of the residual variance of a "well-diversified" portfolio, namely the part of the total variance related to the disturbance term ε only

- for $\mu = 2$, the residual variance goes to zero as $1/N$, so that the market return should be the only relevant explaining factor if the number of traded assets is large enough;
- for $\mu = 1$, the residual variance goes very slowly to zero, so that one can expect a significant contribution to the total risk and a strong impact of the Zipf factor z for large (but finite) market sizes;
- for $\mu = 1/2$, the residual variance does not go to zero and one can expect that the contribution of the residual variance to the total risk remains a finite contribution as the size of the market increases without bounds.

For each value $\mu = 2$, $\mu = 1$ and $\mu = 1/2$, we generate 100 synthetic markets of each size $N = 1,000$ and $N = 10,000$. For each market, we construct 20 equally weighted portfolios (randomly drawn from each market) so that each of the 20 equally-weighted portfolios is made of 1,000/20=50 assets and 10,000/20=500 assets when $N = 1,000$ and $N = 10,000$, respectively. We regress their returns on the returns of the market portfolio (r_m), on the returns of the market portfolio and of the Zipf factor (r_m, z), on the returns of the market portfolio and of the (overall) equal-weighted portfolio (r_m, r_e), on the returns of the market portfolio and of an arbitrary under-diversified portfolio (r_m, r_u), and on the returns of the market portfolio and of an arbitrary well-diversified arbitrage portfolio (r_m, r_a). Using the 100 market simulations for each case (μ, N), Table 4 summarizes the mean, minimum and maximum values of the coefficient of determination R^2 of these five regressions of the 20 equally weighted portfolios.

Notice that our model implies according to (16) or (18) that the factor z is correlated with the residuals. In the regressions, this is not the case. Therefore, it is *a priori* something to be concerned about. However, as shown by the results in Table 4, the regression by OLS works quite well.

For $\mu = 2$, as was expected, the market return is the only relevant factor: it accounts on average for about 95% and 99% (for $N = 1,000$ and $N = 10,000$ assets, respectively) of the total variance of the 20 equally-weighted portfolios under considerations. The fact that the explained variance increases from 95% to 99% when going from $N = 1,000$ to $N = 10,000$ assets, results from the standard diversification effect since each portfolio has more assets. The minimum and maximum values of the R^2 remains very close to their respective mean values.

For $\mu = 1$, the market factor explains a much smaller part of the total variance compared with the previous case (80% and 88%, respectively for $N = 1,000$ and $N = 10,000$ assets). As expected, the lack of explanatory power of the market factor is stronger for the markets with the smallest number $N = 1,000$ of traded assets. In addition, the minimum R^2 (1% and 20%, resp.) departs strongly from its mean value. Besides, the regression on the market factor and the Zipf factor z (which is readily accessible in the case of a numerical simulation) provides a level of explanation (95% and 99%, respectively) comparable to that of the case $\mu = 2$ for which full diversification of the residual risk occurs. Moreover, the equally-weighted portfolio provides the same level of explanation as z itself. This is particularly interesting insofar as z is not observable in a real market while the return on the equally-weighted portfolio can always be calculated, or at least proxied. We find more generally that any well-diversified portfolio provides overall the same explaining power. This result is simply related to the fact that the Zipf factor z is responsible for the lack of diversification of "well-diversified" portfolios (when μ is less than but approximatly 1) so that the return on any "well-diversified" portfolio p reads $r_p \simeq \alpha_p + \beta_p \cdot r_m + \mathrm{E}[\gamma] \cdot z$. This suggests that the equally-weighted portfolio or any well-diversified portfolio, in so far as it is strongly sensitive to the Zipf factor z, may act as a good proxy for this factor.

In contrast, the regression on any under-diversified portfolio, while improving on the regression performed just using the market portfolio, remains of lower quality: the gain in R^2 is only 5-6% on average with respect to the regression on the market portfolio alone. Finally, table 4 shows that the introduction of an arbitrage portfolio does not improve the regression. This is due to the fact that arbitrage portfolios are not asymptotically sensitive to the Zipf factor z in the large N limit.

The same conclusions hold qualitatively for synthetic markets generated with $\mu = 1/2$, with the important quantitative change that the explanatory power of the market factor does not increase with the market size N. This expresses the predicted property that the Zipf factor z should have an asymptotically finite contribution to the residual variance as the size of the market increases without bounds.

Finally, our numerical tests confirm that the distributional properties of the γ's (the factor loading of the residuals on the Zipf factor z) have no significant impact on the results of the simulation, provided that $\mathrm{E}\left[|\gamma|\right] < \infty$.

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Table 1: Summary statistics

Mean value, standard deviation and correlation coefficients of the monthly returns on the market portfolio (excess return over the one month T-bill), on the equally weighted portfolio (excess return over the one month T-bill), on the ICC factor (spread between the return on the equally weighted portfolio and the market portfolio), on the SMB and the HLM factors over the time period from January 1927 to December 2005.

				Correlation			
	Mean	Std	Sharpe	Re	Zipf	SMB	HML
Rm	0.64%	5.48%	0.12	0.90	0.38	0.33	0.22
Re	1.03%	7.52%	0.14		0.74	0.63	0.35
Zipf	0.39%	3.48%	0.11			0.86	0.42
SMB	0.25%	3.37%	0.07				0.09
HML	0.41%	3.60%	0.11				

Table 2: Empirical results for size and book-to-market portfolios

Tests of three asset pricing models – the market model, the Zipf model, and the Fama-French model – with 25 value-weighted portfolios sorted on size and book to market. Data from July 1931 to December 2005 (894 months).

	Low	2	3	4	High	Low	2	3	4	High	Low	2	3	4	High
Panel A: Market model															
	α					$t(\alpha)$					R^2				
Small	-0.52	-0.06	0.25	0.46	0.59	-1.86	-0.24	1.47	2.77	3.01	0.53	0.57	0.67	0.67	0.61
2	-0.14	0.18	0.34	0.40	0.44	-0.96	1.45	2.99	3.31	2.86	0.72	0.78	0.78	0.76	0.72
3	-0.10	0.19	0.26	0.33	0.33	-0.89	2.43	3.11	3.36	2.37	0.82	0.86	0.86	0.81	0.76
4	-0.01	0.06	0.21	0.25	0.19	-0.09	0.91	2.73	2.56	1.33	0.87	0.91	0.87	0.83	0.76
Big	-0.04	0.01	0.06	0.01	0.19	-0.69	0.18	0.78	0.07	1.38	0.92	0.91	0.85	0.79	0.68
Mean Abs			0.22					1.75					0.77		
Panel B: Zipf model (Market + Zipf)															
			α					$t(\alpha)$					R^2		
Small	-0.88	-0.44	-0.05	0.14	0.21	-3.85	-2.75	-0.47	1.58	2.04	0.68	0.80	0.87	0.91	0.89
2	-0.32	-0.04	0.14	0.19	0.17	-2.81	-0.46	1.92	2.42	1.73	0.81	0.91	0.91	0.90	0.88
3	-0.24	0.09	0.15	0.19	0.14	-2.78	1.36	2.19	2.49	1.23	0.88	0.91	0.90	0.88	0.85
4	-0.02	0.00	0.14	0.15	0.03	-0.33	0.00	1.97	1.73	0.21	0.87	0.92	0.89	0.86	0.82
Big	0.02	0.05	0.06	-0.04	0.11	0.41	0.94	0.89	-0.36	0.79	0.94	0.92	0.85	0.80	0.71
Mean Abs			0.16					1.51					0.86		
Panel C: Fama-French model (Market + SMB + HML)															
			α					$t(\alpha)$					R^2		
Small	-0.89	-0.45	-0.10	0.05	0.06	-3.77	-3.03	-1.03	0.68	0.69	0.67	0.82	0.89	0.93	0.93
2	-0.21	-0.05	0.08	0.07	-0.01	-2.49	-0.78	1.32	1.34	-0.19	0.90	0.94	0.94	0.95	0.96
3	-0.15	0.09	0.08	0.08	-0.07	-2.31	1.48	1.25	1.26	-1.03	0.93	0.92	0.93	0.93	0.94
4	0.08	-0.03	0.07	0.01	-0.21	1.45	-0.45	1.07	0.12	-2.59	0.93	0.92	0.91	0.92	0.93
Big	0.07	0.04	-0.02	-0.22	-0.09	1.89	0.86	-0.29	-3.47	-0.90	0.95	0.92	0.90	0.93	0.83
Mean Abs			0.13					1.43					0.91		

Table 3: Empirical results for industry portfolios

Tests of three asset pricing models – the market model, the Zipf model, and the Fama-French model – with 30 value-weighted portfolios sorted on industry. Data from July 1931 to December 2005 (894 months).

	Market model			Zipf model			Fama-French model		
	α	$t(\alpha)$	R^2	α	$t(\alpha)$	R^2	α	$t(\alpha)$	R^2
Food	0.22	2.63	0.70	0.25	2.93	0.71	0.22	2.61	0.71
Beer	0.34	2.00	0.49	0.29	1.69	0.51	0.26	1.56	0.51
Smoke	0.46	2.99	0.34	0.49	3.19	0.35	0.45	2.92	0.35
Games	-0.10	-0.59	0.69	-0.21	-1.37	0.73	-0.20	-1.26	0.72
Books	-0.06	-0.46	0.69	-0.12	-0.97	0.71	-0.14	-1.14	0.71
Hshld	0.08	0.72	0.68	0.11	0.97	0.68	0.10	0.93	0.68
Clths	0.06	0.42	0.50	-0.02	-0.14	0.54	0.02	0.18	0.56
Hlth	0.26	2.36	0.65	0.30	2.73	0.66	0.33	3.05	0.66
Chems	0.09	0.95	0.78	0.12	1.33	0.79	0.11	1.17	0.79
Txtls	-0.10	-0.70	0.67	-0.23	-1.85	0.74	-0.28	-2.26	0.75
Cnstr	-0.08	-0.94	0.84	-0.13	-1.47	0.85	-0.14	-1.61	0.85
Steel	-0.16	-1.18	0.75	-0.23	-1.70	0.76	-0.30	-2.30	0.77
FabPr	-0.03	-0.33	0.85	-0.08	-0.95	0.86	-0.09	-1.08	0.86
ElcEq	0.11	1.01	0.82	0.12	1.16	0.82	0.12	1.09	0.82
Autos	-0.02	-0.15	0.71	-0.06	-0.42	0.72	-0.11	-0.80	0.72
Carry	0.07	0.53	0.70	0.00	0.03	0.72	-0.06	-0.49	0.73
Mines	0.09	0.51	0.46	0.03	0.17	0.47	0.01	0.05	0.48
Coal	0.43	1.69	0.24	0.38	1.48	0.25	0.35	1.38	0.25
Oil	0.25	2.03	0.59	0.28	2.25	0.59	0.21	1.70	0.62
Util	0.10	0.82	0.58	0.10	0.87	0.58	0.01	0.12	0.63
Telcm	0.12	1.22	0.58	0.14	1.47	0.59	0.16	1.66	0.59
Servs	0.39	1.65	0.26	0.37	1.56	0.26	0.52	2.26	0.32
BusEq	0.14	1.20	0.71	0.15	1.23	0.71	0.29	2.79	0.78
Paper	0.12	1.24	0.74	0.13	1.39	0.74	0.12	1.19	0.74
Trans	-0.10	-0.84	0.73	-0.18	-1.56	0.76	-0.29	-2.72	0.80
Whlsl	-0.18	-1.21	0.62	-0.29	-2.07	0.67	-0.26	-1.88	0.68
Rtail	0.10	1.05	0.73	0.10	1.02	0.73	0.14	1.38	0.74
Meals	0.16	1.16	0.60	0.11	0.80	0.61	0.13	0.99	0.62
Fin	0.01	0.14	0.84	-0.01	-0.08	0.84	-0.07	-0.78	0.85
Other	-0.14	-1.17	0.70	-0.19	-1.65	0.71	-0.16	-1.36	0.72
Mean Abs	0.15	1.13	0.64	0.17	1.35	0.66	0.19	1.49	0.67

Table 4: Numerical simulations

Average, minimum and maximum value of the R^2 of the regression of the return of 20 equally weighted portfolios (randomly drawn from a market of $N = 1000$ and $N = 10,000$ assets according to the model (17)) on the market portfolio (r_m), on the market portfolio and the internal consistency factor (r_m, f), on the market portfolio and the (overall) equally weighted portfolio (r_m, r_e), on the market portfolio and an under-diversified portfolio (r_m, r_u) and on the market portfolio and a well-diversified arbitrage portfolio (r_m, r_a). Different market situations are considered with distributions of firm sizes with tail index μ which varies from 0.5 to 2.

		N=1000					N=10,000				
		r_m	r_m, f	r_m, r_e	r_m, r_u	r_m, r_a	r_m	r_m, f	r_m, r_e	r_m, r_u	r_m, r_a
$\mu = 2$	Mean	94%	94%	95%	94%	94%	99%	99%	99%	99%	99%
	Min	90%	93%	93%	90%	90%	99%	99%	99%	99%	99%
	Max	96%	96%	96%	96%	96%	100%	100%	100%	100%	100%
$\mu = 1$	Mean	80%	95%	95%	86%	82%	88%	99%	99%	93%	89%
	Min	1%	91%	91%	42%	17%	20%	99%	99%	66%	20%
	Max	95%	100%	100%	95%	95%	99%	100%	100%	99%	99%
$\mu = 1/2$	Mean	56%	97%	97%	79%	64%	56%	100%	100%	83%	63%
	Min	2%	89%	89%	34%	15%	1%	96%	97%	15%	3%
	Max	100%	100%	100%	100%	100%	100%	100%	100%	100%	100%

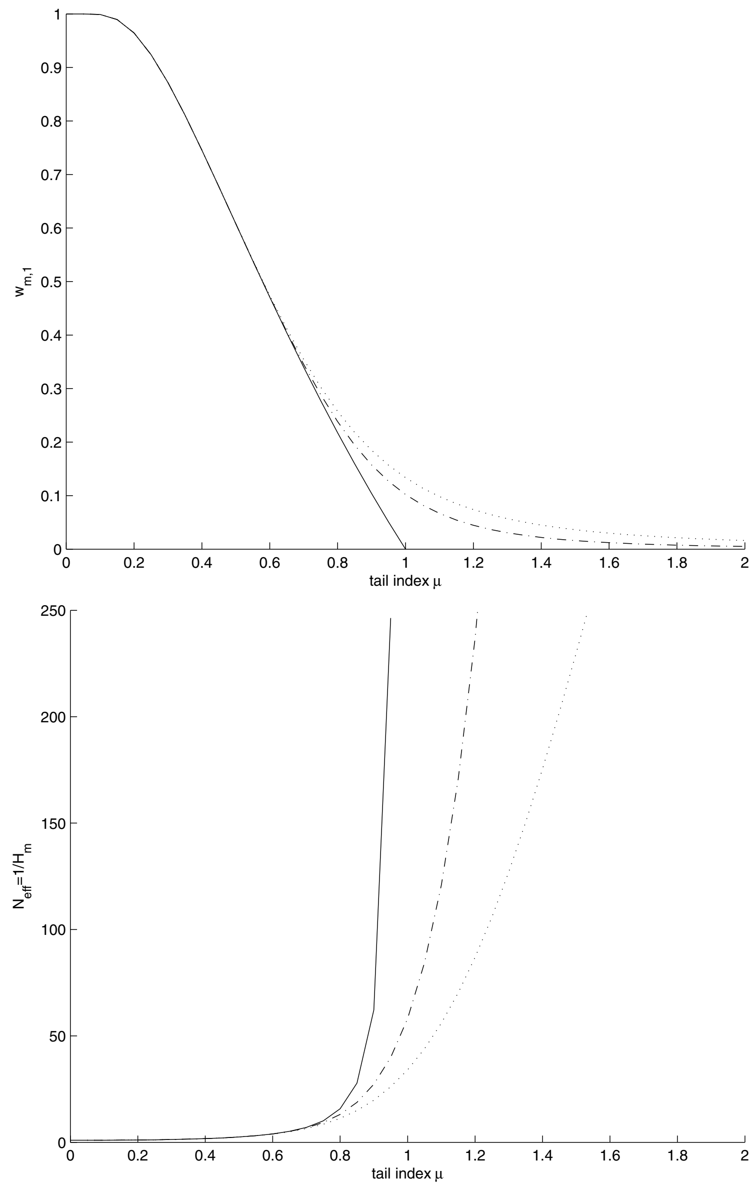

Figure 1: **Concentration of the market portfolio**. The upper panel shows the weight of the largest firms in the market portfolio as a function of the tail index μ of the Pareto distribution of firm sizes. The lower panel shows the inverse of the Herfindahl index of the market portfolio – namely the effective number of assets N_{eff} in the market portfolio – as a function of the tail index μ of the Pareto distribution of firm sizes. In both cases, the continuous line provides the values in the limit of an infinite economy while the dotted and dash-dotted curves correspond to the cases of an economy with one thousand and ten thousand firms respectively.